Distributed Optimization for Combinatorial Coding Subgraph Selection Problems

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***Abstract*—Many network design cases with network coding can be formulated as combinatorial optimization problems, which are NP-hard and hard to approximate even in a centralized manner. In this paper, guided by Markov approximation framework, we provide a unified distributed solution for one important subcategory of combinatorial network coding problems: combinatorial coding subgraph selection problems. We show the scheme by studying one problem instance: optimal coding subgraph selection over wireline networks under arbitrary bounds on graph diameter [1]. We develop a distributed Markov chain based solution. We obtain analytical results for the impacts of measurement errors on the designed Markov chain. We also discuss the trade-off between the approximation gap and the mixing time of the designed Markov chain. We emphasis that though the analysis is quite involved, the resulting distributed solution is actually simple to implement.**

Keywords-Distributed Optimization; Coding Subgraph Selection; Markov Approximation Framework; Markov Chain; Mixing Time

I. Introduction

Network coding, introduced in [2], allows intermediate nodes to perform coding operation instead of routing (store and forward). In general, when network coding is applied for network design, there are two key steps: subgraph selection and code construction. Subgraph selection is the selection of network resources by choosing links, nodes and corresponding link service rates to support a transmission session. The selection criterions include maximizing system performance such as network utility or throughput, and minimize system cost such as interference, energy or latency.

Once the coding subgraph is established, we perform network coding, *i.e.*, the construction of network codes for the given subgraph. Despite rich results on network coding [3, 4], there are still a lot of challenges when we try to solve combinatorial network coding problems, *i.e.*, combinatorial network design problems with network coding. In general, combinatorial network coding problems are NP-hard and hard to approximate even in a centralized manner. The hardness of combinatorial explosion for these problems comes from the following aspects:

- **Coding Subgraph**. The number of feasible coding subgraphs can be exponentially large. For example. the coding subgraphs are composed by possible subsets of all available paths, trees or non-interfering wireless links.
- **Code Construction**. The number of feasible codes can be exponentially large. For example, some linear code constructions for multi-session network coding [4].

In this paper, we do not discuss the combinatorial code construction problem. Instead, we adopt random linear network codes [4]. Based on this coding scheme, the following fundamental question remains open:

- How to solve combinatorial coding subgraph selection problem in a distributed manner?

In this paper, we answer the above question and make the following contributions:

- We solve the combinatorial coding subgraph selection problem in a unified way. Our distributed solutions are inspired by a set of log-sum-exp approximation and Markov chain based arguments expounded in [5]. Further, our distributed solutions are simple to implement, while having performance guarantees.
- We provide a distributed solution for one typical case of combinatorial coding subgraph selection problem: optimal coding subgraph selection over wireline networks under arbitrary bounds on graph diameter. Our solution optimizes the selection of multicast trees (coding subgraph) and achieves multicast rates arbitrarily close to optimum.
- We give upper bounds on the gap to optimum when the transition rates of the designed Markov chain contain errors. We also discuss the trade-off between approximation gap and the mixing time of the designed Markov chain.

Due to page limits, all proofs are omitted here and details can be found in our technical report [6].

II. General Combinatorial Coding Subgraph Selection Problem

Consider a network modeled as a graph G, we denote $\mathcal{F}$ as the set of all feasible coding subgraphs. For a feasible subgraph $f \in \mathcal{F}$, let ϕ_f be the system performance under f, including throughput, power, and delay as special cases. The problem of maximizing the system performance by choosing the best coding subgraph can then be cast as following combinatorial optimization problem called combinatorial coding subgraph selection problem(CCSS):

$$\mathbf{CCSS}: \max_{f \in \mathcal{F}} \phi_f \tag{1}$$

.

In general, problem **CCSS** is computational intractable because the number of all feasible coding subgraphs can be

[1] A graph's diameter is the largest number of nodes which must be traversed in order to travel from one node to another when paths which backtrack, detour, or loop are excluded from consideration [1].

exponentially large. Further, there is no effective approximate solution to the problem even in a centralized manner

In practice, it is often acceptable to solve the problem approximately with a guaranteed bound, but in a distributed manner. Systems running distributed algorithms are more robust to user and system dynamics than those running centralized algorithms.

In this paper, we adopt Markov approximation framework proposed in [5], which enable us to design distributed scheduling algorithms with provable optimality of network performance. The essence of this framework is that, when using the log-sum-exp function to approximate the optimal value of any combinatorial problem, we end up with a solution that can be interpreted as the stationary probability distribution of a class of time-reversible Markov chains. Certain carefully designed Markov chains among this class yield distributed algorithms that solve the log-sum-exp approximated combinatorial network optimization problem.

1) Log-sum-exp Approximation: First, we use the *log-sum-exp* function to approximate the *max* function smoothly, shown as follows:

$$\max_{f\in\mathcal{F}} \phi_f \approx \frac{1}{\beta}\log(\sum_{f\in\mathcal{F}}\exp[\beta\,\phi_f]) \tag{2}$$

where β is a positive constant. Let $|\mathcal{F}|$ denote the size of the set $\mathcal{F}$, then the approximation accuracy is known as follows [5]:

Proposition 1.

$$0 \le \frac{1}{\beta}\log(\sum_{f\in\mathcal{F}}\exp[\beta\,\phi_f]) - \max_{f\in\mathcal{F}}\phi_f \le \frac{1}{\beta}\log|\mathcal{F}| \tag{3}$$

As $\beta\to\infty$, the approximation gap disappears.

We have some important observations in the following proposition [5].

Proposition 2. $\frac{1}{\beta}\log(\sum_{f\in\mathcal{F}}\exp[\beta\,\phi_f])$ *is the optimal value of the following optimization problem*

$$\mathbf{CCSS}-\beta:\ \max_{\boldsymbol{p}\ge 0}\ \sum_{f\in\mathcal{F}} p_f\phi_f - \frac{1}{\beta}\sum_{f\in\mathcal{F}} p_f\log p_f \tag{4}$$
$$s.t.\ \sum_{f\in\mathcal{F}} p_f = 1.$$

and the corresponding optimal solution is

$$p_f^*(\boldsymbol{\phi}) = \frac{\exp[\beta\phi_f]}{\sum_{f'\in\mathcal{F}}\exp[\beta\phi_{f'}]},\ \forall f\in\mathcal{F}, \tag{5}$$

where $\boldsymbol{\phi}\triangleq[\phi_f, f\in\mathcal{F}]$.

By time-sharing among different configurations f according to their portions $p_f^*(\boldsymbol{\phi})$, we can solve the problem $\mathbf{CCSS}-\beta$. Note that an equivalent formulation of problem $\mathbf{CCSS}$ is

$$\mathbf{CCSS2}:\ \max_{\boldsymbol{p}\ge 0}\ \sum_{f\in\mathcal{F}} p_f\phi(f) \tag{6}$$
$$\text{s.t.}\ \sum_{f\in\mathcal{F}} p_f = 1,$$

where p_f is the time fraction (or probability) of the feasible coding subgraph f. Therefore, by the *log-sum-exp* approximation in (2), we are implicitly solving an approximated version of the problem $\mathbf{CCSS2}$, off by an *entropy* term $-\frac{1}{\beta}\sum_{f\in\mathcal{F}} p_f\log p_f$.

2) Distributed Markov Chain Monte Carlo: To obtain $p_f^*(\boldsymbol{\phi}), f\in\mathcal{F}$ (5) in a distributed way, we construct a time-reversible continuous-time Markov chain with its state space being $\mathcal{F}$ and its stationary distribution being $p_f^*(\boldsymbol{\phi}), f\in\mathcal{F}$. Denote $q_{f,f'}$ as the non-negative transition rate between two states f and f', it suffices to design $q_{f,f'}$ in a distributed manner such that

- the resulting Markov chain is irreducible, *i.e.*, any two states are reachable from each other,
- and the detailed balance equation is satisfied: for all f and f' in $\mathcal{F}$ and $f\neq f'$, $p_f^*(\boldsymbol{\phi})q_{f,f'} = p_{f'}^*(\boldsymbol{\phi})q_{f',f}$,, *i.e.*, $\exp(\beta\phi_f)q_{f,f'} = \exp(\beta\phi_{f'})q_{f',f}$.

We have two degrees of freedom in designing a time-reversible Markov chain distributedly:

- **The state space structure**: two states f, f' have one edge connecting them if and only if the direct transitions between f and f' involves only local information
- **Transition rate**: the transition rates $q_{f,f'}$ and $q_{f',f}$ utilize only local information of f and f'.

In practice, however, the designed transition rates may need to utilize global information that is hard to obtain. Therefore, we use the locally available information as the estimation of the global information. On the other hand, even if the designed transition rates utilize only local information, this local information can be imprecise due to noisy measurements. All above factors lead to inaccurate transition rates, which further lead to inaccurate stationary distributions instead of the designed stationary distribution (5).

To study the impacts of inaccurate transition rates, we can view the Markov chain formed by the inaccurate transition rates as a perturbed version of the designed Markov chain with exact transition rates. Under some mild conditions [7], we can obtain upper bounds on the differences between inaccurate and exact stationary distributions.

III. Optimal Multicast Rate of Wireline Networks with Network Coding and Delay Concern

In this Section, we focus on one problem instance: optimal coding subgraph selection over wireline networks under arbitrary bounds on graph diameter.

Existing coding subgraph selection algorithms including back-pressure algorithms and convex optimization algorithm [4, 8] are not suitable for networks with bounds on graph diameter. The reason is that these algorithms tend to explore all available links in the network and need to maintain very long end-to-end paths between sources and destinations. Instead, we utilize multiple multicast trees to multicast data packets, where the depth of each tree is bounded and the number of simultaneously used trees is limited. In this way, the requirement on graph diameter bound is satisfied. The network coding is operated over the subgraph composed by the selected multicast trees. We aim to achieve maximum multicast rate by selecting the best coding subgraph.

Next we introduce the problem formulation.

A. Problem Formulation

We consider a wireline network $G = (V, L)$, where V is the set of nodes and L is the set of links. Each link $l \in L$ is associated with a capacity C_l.

We focus on a single-source multicast scenario, *i.e.*, the single source s multicasts data packets to its destination set R_s. Let J_s denote the set of multicast trees (Steiner tree) available for source s, where the depth of each tree $j \in J_s$ is bounded by a positive constant B_j. We allow different trees to have different depth bounds. Due to limited system resource or overhead concern, source s selects at most D_s trees from J_s.

We denote $\mathcal{F}$ as the set of all feasible tree configurations over G under tree depth bounds. Each tree configuration $f \in \mathcal{F}$ is defined as a set of multicast trees simultaneously used by source s. Let $|f|$ denote the size of configuration f, then we have $|f| = D_s$.

For a feasible configuration $f \in \mathcal{F}$, let ϕ_f be the maximum multicast rate under f. ϕ_f is obtained by network coding, where the coding occurs on all overlapping segments of different trees of source s. In this sense, f can be regarded as a coding subgraph [8]. Then the problem of achieve maximum multicast rate by selecting the best coding subgraph is formulated as follows:

$$\mathbf{MMR}: \max_{f \in \mathcal{F}} \phi_f \tag{7}$$

In general, problem $\mathbf{MMR}$ is computational intractable and is hard to approximate even in a centralized manner.

In this Section, we address this problem by providing a distributed solution. In particular, we first develop a distributed multicast algorithm that can achieve ϕ_f under arbitrary $f \in \mathcal{F}$, then we design a distributed algorithm that selects the best tree configuration (coding subgraph).

B. The Distributed Multicast Algorithm

Given a feasible tree configuration $f \in \mathcal{F}$, ϕ_f is obtained by solving the following utility maximization problem:

$$\mathbf{UM}: \max_{\boldsymbol{x} \geq 0} \quad U(\sum_{j \in f} x_j) \tag{8}$$

$$\text{s.t.} \quad x_j \leq C_l, \forall j \in f, l \in j, , l \in L \tag{9}$$

where $U(\cdot)$ is a twice-differentiable, increasing and strictly concave function, $x_j, j \in f$ is the rate of multicast tree j under f, and $l \in j$ means that tree j uses the link l. The constraint (9) comes from the flow-sharing property of network coding [4].

Let the optimal solution of problem $\mathbf{UM}$ denoted by $\hat{x}_j, j \in f$, then ϕ_f is given by

$$\phi_f = \sum_{j \in f} \hat{x}_j \tag{10}$$

Since the objective function in (8) is not strictly concave in $x_j, j \in f$, the associated algorithms solving $\mathbf{UM}$ based on duality may have the oscillation problem [9]. To circumvent the difficulty due to the lack of strict concavity, we use proximal optimization method [9]. With this method, we enforce strict concavity by adding a quadratic term to the objective function and then iterate to eliminate the effects of the term.

Due to space limits, we omits details on distributed multicast algorithm, which is based on proximal optimization method, Lagrange dual decomposition and random linear network coding.

C. The Distributed Coding Subgraph Selection Algorithm

Now given any $f \in \mathcal{F}$, we can obtain ϕ_f distributedly. Next, we apply the generic way developed in Section II to solve problem $\mathbf{MMR}$ (7) in a distributed way.

By log-sum-exp approximation, we solve an approximated problem instead:

$$\mathbf{MMR} - \beta: \max_{\boldsymbol{p} \geq 0} \sum_{f \in \mathcal{F}} p_f \phi_f - \frac{1}{\beta} \sum_{f \in \mathcal{F}} p_f \log p_f \tag{11}$$

$$\text{s.t.} \sum_{f \in \mathcal{F}} p_f = 1.$$

The corresponding optimal solution is

$$p_f^*(\boldsymbol{\phi}) = \frac{\exp[\beta \phi_f]}{\sum_{f' \in \mathcal{F}} \exp[\beta \phi_{f'}]}, \ \forall f \in \mathcal{F}, \tag{12}$$

where $\boldsymbol{\phi} \triangleq [\phi_f, f \in \mathcal{F}]$.

Now we construct a time-reversible continuous-time Markov chain with its state space being $\mathcal{F}$ and its stationary distribution being $p_f^*(\boldsymbol{\phi}), f \in \mathcal{F}$.

First, we set the transition rate $q_{f,f'}$ between two configurations f and f' to be zero, unless f and f' satisfy that

C1: $|f \cup f' - f \cap f'| = 2$;

C2: $f \cup f' - f \cap f' \in J_s$.

In this way, the transition from f to f' corresponds to the source s switching a single multicast tree.

Second, for f and f' satisfying C1 and C2, we choose $q_{f,f'}$ and $q_{f',f}$ as follows:

$$q_{f,f'} = \frac{\theta}{\exp(\beta \phi_f)} \tag{13}$$

$$q_{f',f} = \frac{\theta}{\exp(\beta \phi_{f'})} \tag{14}$$

where θ is a positive constant.

So to implement transition rates (13) and (14), the source s need to collect information of ϕ_f and $\phi_{f'}$. This can be done locally since source s can measure its multicast rate. We briefly describe the distributed implementation as follows.

Stag1: Initially, source s randomly selects D_s trees from its available multicast tree set J_s.

Stag2: Source s measures the value of ϕ_f, where f is the current configuration. Then source s counts down according to a random number following an exponential distribution with parameter $\frac{D_s(|J_s|-D_s)\theta}{\exp(\beta \phi_f)}$.

Stag3: When the count-down expires, source s randomly switches one in-use tree in f with one remaining candidate in $J_s - f$. Then the source s returns to **Stag1**.

We establish the following result:

Proposition 3. *This implementation in fact realizes a time-reversible Markov chain with stationary distribution in* (12)*.*

D. The Overall Distributed Solution

By combining the distributed multicast algorithm and the distributed coding subgraph selection algorithm and operating them in tandem, we obtain the overall distributed solution. Now we discuss the convergence of the overall distributed solution.

In practice, however, our measurements are always accompanied with errors, which leads to the inaccurate values of $\phi(f)$ for any configuration $f \in \mathcal{F}$. By (13), we know that the transition rates of Markov chain are also inaccurate. Therefore, the designed Markov chain with inaccurate transition rates does *not* converge to the desired stationary distribution $p_f^*, f \in \mathcal{F}$ (12). Next, we characterize the gap to the desired distribution and the gap to the optimal multicast rate.

E. Impact of Inaccurate Transition Rates

We adopt the following settings: (1) for each configuration $f \in \mathcal{F}$ with ϕ_f, errors (deviations from ϕ_f) belong to the bounded region$[-\Delta_f, \Delta_f]$. These errors are quantized into $2n_f + 1$ levels: $\{\frac{j}{n_f}\Delta_f, j = -n_f, \ldots, n_f\}$. Therefore, the maximum multicast rate under f obtained by measurement belongs to the set: $\{\phi_f + \frac{j}{n_f}\Delta_f, j = -n_f, \ldots, n_f\}$.

We denote the stationary distribution with accurate and inaccurate transition rates as $\boldsymbol{p^*} : (p_f^*, f \in \mathcal{F})$ and $\bar{\boldsymbol{p}} : (\bar{p}_f, f \in \mathcal{F})$ respectively. To characterize the gap between $\boldsymbol{p^*}$ and $\bar{\boldsymbol{p}}$, we adopt one common measure: total variance distance, which is denoted by $d_{TV}(\boldsymbol{p^*}, \bar{\boldsymbol{p}}) \triangleq \frac{1}{2}\sum_{f\in\mathcal{F}} |p_f^* - \bar{p}_f|$. By following the method in [7], we obtain the following result:

Theorem 1. *The upper bound of $d_{TV}(\boldsymbol{p^*}, \bar{\boldsymbol{p}})$ is shown as follows:*

$$d_{TV}(\boldsymbol{p^*}, \bar{\boldsymbol{p}}) \leq 1 - \exp(-2\beta\Delta_{\max}). \tag{15}$$

where $\Delta_{\max} = \max_{f\in\mathcal{F}} \Delta_f$ and $\phi_{\max} = \max_{f\in\mathcal{F}} \phi(f)$. Further, the gap between maximum multicast rate with $\boldsymbol{p^*}$ and maximum multicast rate with $\bar{\boldsymbol{p}}$ is no more than $2\phi_{\max}(1 - \exp(-2\beta\Delta_{\max}))$.

F. Trade-off Between Approximation Gap and Mixing Time of Markov Chain

In this subsection, we study the trade-off between approximation gap and mixing time of Markov chain. This trade-off is parameterized by β.

First, we investigate the impact of β on the approximation gap. We assume the optimization solution to the problem **MMR** (7) is unique. The extension to multiple optimization solutions is straightforward. We denote this unique optimization as f_0 and $f_0 = \arg\max_{f\in\mathcal{F}} \phi_f$. Then the corresponding probability distribution of tree configurations (coding subgraphs) is $\boldsymbol{\pi} : (\pi_f, f \in \mathcal{F})$:

$$\pi_f = \begin{cases} 1 & \text{if } f = f_0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

We define the approximation gap as

$$d_{TV}(\boldsymbol{p^*}, \boldsymbol{\pi}) \triangleq \frac{1}{2}\sum_{f\in\mathcal{F}} |p_f^* - \pi_f|, \tag{17}$$

i.e., the total variance distance between $\boldsymbol{p^*}$ and $\boldsymbol{\pi}$. Then we have

Theorem 2. *The approximation gap $d_{TV}(\boldsymbol{p^*}, \boldsymbol{\pi})$ is a decreasing function of β. As $\beta \to \infty$, the approximation gap is zero.*

Next, we investigate the impact of β on the mixing time of the designed Markov chain. Recall that $\boldsymbol{p^*}$ is the stationary distribution of the designed Markov chain. Let $\boldsymbol{H}_t(f)$ denote the probability distribution of all states in $\mathcal{F}$ at time t given that the initial state is f. We define mixing time as follows:

$$t_{mix}(\epsilon) \triangleq \inf\left\{t \geq 0 : \max_{f\in\mathcal{F}} d_{TV}(\boldsymbol{H}_t(f), \boldsymbol{p^*}) \leq \epsilon\right\} \tag{18}$$

Then we have

Theorem 3. *The mixing time $t_{mix}(\epsilon)$ is upper bounded as follows:*

$$t_{mix}(\epsilon) \leq \frac{|\mathcal{F}|^2 \cdot D_s(|J_s| - D_s) \cdot \exp(4\beta\phi_{max})}{\theta \cdot \frac{1}{\ln\frac{|\mathcal{F}|}{4\epsilon^2} + \beta\phi_{\max}}} \tag{19}$$

This upper bound is an increasing function of β.

IV. Conclusions

In this paper, we develop a unified approach for combinatorial coding subgraph selection problems. We show the method by studying a special case: optimal coding subgraph selection over wireline networks under arbitrary bounds on graph diameter. We design a distributed solution including a distributed multicast algorithm achieving maximum multicast rate under arbitrary tree configuration, and a distributed Markov chain based algorithm to optimize the selection of multicast trees. We characterize the gap to the optimum distribution with inaccurate transition rates of the designed Markov chain. We also discuss the trade-off between the approximation gap and the mixing time of the designed Markov chain.

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