Online Algorithms for Automotive Idling Reduction with Effective Statistics

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***Abstract*—Idling, or running the engine when the vehicle is not moving, accounts for 13% - 23% of vehicle driving time and costs billions of gallons of fuel each year. In this paper, we consider the problem of idling reduction under the uncertainty of vehicle stop time. We abstract it as a classic ski rental problem, and propose a constrained version with two statistics μ_{B^-} and q_{B^+}, the expected length of short stops and the probability of long stops. We develop online algorithms that combine the best of the well-known deterministic and randomized schemes to minimize the worst case competitive ratio. We demonstrate the robustness of the algorithms in terms of both worst case guarantee and average case performance using simulation and real-world driving data.**

***Index Terms*—Automotive Idling Reduction, Online Algorithm, Ski Rental Problem, Competitive Analysis**

I. Introduction

FUEL economy has become a major concern in vehicle designs, due to its significant environmental impact and the foreseeable shortage of fossil oil. There is an enormous amount of efforts in place to reduce the vehicle fuel consumption and emission. For example, the US Environmental Protection Agency [1] imposes the CO_2 emission standard of Model Year 2014 to be 315 grams/mile for passenger cars. In 2025, this number has to be decreased to 163 grams/mile, or equivalently 54.5 miles per gallon (mpg), in parity with today's Prius (around 51 mpg). This greatly motivates the development and commercialization of electric vehicles, hybrid electric vehicles, and other energy efficient vehicles.

In this paper, we consider the problem of **reducing the cost associated with vehicle idling**. An idling vehicle runs its engine when it is not moving, which causes unnecessary waste of fuel. The average amount of idling has been measured at 13% to 23% of the total vehicle operating time, according to surveys conducted in North America and Europe [2]. In US alone, idling vehicles use more than 6 billion gallons of fuel at a cost of more than $20 billion each year [3]. These (possibly astonishing) facts have triggered significant legislation efforts against unnecessary long idling. For example, Toronto City Council at its meeting on July 8, 2010, made changes to the Idling Control By-Law, to impose an idling limit of 1 minute [4]. Similar rules and laws can be found throughout US [5] and Europe [6].

In order to reduce the cost associated with vehicle idling (including fuel and emissions), the driver may manually turn off the engine, when he/she expects to experience a long stop. Alternatively, Stop-Start Systems (SSS) have been proposed to automatically perform the task. Such a system is a key building block in hybrid electric vehicles (HEV), but it can also be added as a new feature to conventional vehicles (those equipped with an internal combustion engine only). In the later case, they are typically referred to as Stop-Start Vehicles (SSV). SSV would turn off the engine immediately when the car stops, and restart the engine when the driver pushes the gas pedal to go forward. Other functions like accessories and lighting are powered by an electrical source other than the vehicle's alternator. In HEV, the strategy can be more complicated, and is out of the scope of this work.

As in the case of idling, restarting the engine also comes with a cost. It is estimated that the fuel consumption for restarting the engine once is equivalent to keeping the engine idling for 10 seconds [2], [7]. Considering other costs associated to engine wear and exhaust gas emission, this number goes up to 28 seconds for SSV or 47 seconds for those without SSS (see Section IV for details). Thus, it is not necessarily the best strategy to turn off the engine immediately. Considering fuel consumption alone, it is better to keep the engine running if the vehicle is known to be at rest for less than 10 seconds.

However, the vehicle stop time is unknown or even hard to estimate in many situations, such as at traffic lights or in heavy traffic. Thus, SSV have to make **online decisions**, i.e., without the a-priori knowledge of the vehicle stop time. In this paper, we consider the problem of finding the best online strategy for the stop-start systems. It can also be provided as a driving tip to drivers of vehicles without stop-start systems. In particular, we claim the following **contributions**:

1) We consider the costs of fuel consumption, emission, and engine wear associated with idling and restart. We abstract the problem as a classic ski rental problem, where a break-even value characterizes the trade-off between keeping the vehicle idle and restarting the engine. Thus, existing solutions on the ski rental problem can be directly adopted.

2) We gain insight into the characteristics of the optimal offline algorithm, and propose a constrained ski rental problem by introducing two new statistics μ_{B^-} and q_{B^+}, where μ_{B^-} is the expected length of short stops, and q_{B^+} is the probability of long stops. We derive two online algorithms,

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for the constrained ski rental problem. One is a closed-form solution, **ONAIR** (an ONline algorithm for Automotive Idling Reduction with effective statistics), that provides performance guarantees (bounded worst case expected competitive ratio). The other is an optimal numerical solution, ONAIRE (ONAIR Exact version), which gives the smallest worst case expected competitive ratio under any traffic conditions.

3) We use real-world data to evaluate the proposed online algorithms. ONAIR and ONAIRE outperform other existing strategies in terms of providing lower bound on worst case performance. At the same time, for average case performance, ONAIR and ONAIRE also perform better than other strategies. In addition, the robustness of the proposed strategies in different driving conditions is evaluated using synthetic data.

The rest of the paper is organized as follows. In Section II, we introduce the problem of SSS online strategy and link that to the classic ski rental problem. We also review related works proposed in the context of the ski rental problem. In Section III, to minimize the worst case expected competitive ratio, we solve the constrained ski rental problem analytically to derive a closed form sub-optimal strategy, and also solve the problem numerically to get the optimal solution. The break-even interval is calculated in Section IV. In Section V, we use real-world driving data and synthetic data to validate the performance of the proposed strategies. Finally, the paper is concluded in Section VI.

II. Idling Reduction Problem

When the car has to stop due to traffic or the driver's needs, there are two possible actions that the driver/SSS can take:

– **Keeping the Vehicle Idle**, which wastes fuel to keep the engine running at a relatively low speed, and consequently with exhaust gas emissions. The associated cost is proportional to the vehicle idling time.

– **Turning off the Engine**. In this case, the engine has to be restarted when the driver pushes the gas pedal. Restarting the engine requires a one-time cost due to 1) fuel consumption and related emission; 2) excessive engine wear costs, including those to the starter and starter battery.

Both costs can be calculated by studying the characteristics of the vehicle and the cost to each part (e.g., starter and starter battery). We use $\mathrm{cost}_{\mathrm{idling}\,/s}$ to denote the cost of idling per unit time, and $\mathrm{cost}_{\mathrm{restart}}$ for the one-time cost to restart the engine. The ratio between these two

$$B = \frac{\mathrm{cost}_{\mathrm{restart}}}{\mathrm{cost}_{\mathrm{idling}\,/s}} \tag{1}$$

denotes the amount of idling time such that the total cost for idling is equal to the cost of turning off and restarting the engine. B is called the **break-even interval**, which plays a key role in the algorithm design. The calculation of B for automotive idling is detailed in Section IV, which considers the fuel consumption and other costs associated to engine wear and exhaust gas emission. However, the existing algorithms for the ski rental problem (summarized in Section II-B) and our algorithms apply to any given B.

During the vehicle stop, decision has to be made whether to continue waiting (and keep the engine idle) or to turn off and restart when the driver intends to move forward. If the vehicle stop time y is known in advance, it is easy to figure out the optimal strategy: if y is less than B (informally, the stop is **"short"**), then it is better to keep the engine idle; otherwise (informally, the stop is **"long"**), the driver/SSS should turn off the engine immediately and restart later.

However, the vehicle stop time is naturally random, and in many situations, such as stops at traffic light or in heavy traffic, it is difficult to estimate before hand. The decision has to be made without having the input y, or in an **online** fashion. In contrast, the optimal strategy with the knowledge of y is called the **offline algorithm**. The problem of designing online algorithm to choose between continuing idling (and paying a repeating cost) or paying a one-time restart cost is exactly the topic of the classic **ski rental problem** [8]. In the ski rental problem [8], a skier has to pay $1 for renting skis for one day or pay $B to buy his own. He cannot find out until which day he is still able to ski due to the unpredictable weather condition. Every day when he goes skiing, an online decision can be made on whether to rent (similar to keeping the vehicle idle) or buy (similar to turning off and restarting the engine).

Online algorithm has a broad range of applications, including Snoopy Caching [8] [9], microgrids power generation scheduling [10], financial decision on renting or buying [11], task assignment and scheduling in multicore system [12], system level power management [13] [14], temperature aware energy minimization [15]. In the following, we review the metric to evaluate the performance of the online algorithms and the existing solutions for the ski rental problem.

A. Competitive Analysis

Competitive analysis is a common way to evaluate online algorithms, which compares the cost incurred by the evaluated strategy with the optimal offline algorithm. Optimal offline algorithm knows stop length a priori, so that it can turn the engine off immediately if the stop is long, or keep it idle if the stop is short. For a stop with length y, the cost of the offline algorithm, denoted as $\mathrm{cost}_{\mathrm{offline}}(y)$, is calculated as

$$\mathrm{cost}_{\mathrm{offline}}(y) = \begin{cases} y & 0 \le y < B \\ B & y \ge B \end{cases} \tag{2}$$

The online algorithm (deterministically or randomly) selects the amount of idling time x. We denote the cost of the online algorithm for a selected x and a given y as $\mathrm{cost}_{\mathrm{online}}(x, y)$. Since the vehicle will wait until x, if $y < x$, the cost is y; otherwise, the cost is the amount of idle time x plus the one time restart cost B.

$$\mathrm{cost}_{\mathrm{online}}(x, y) = \begin{cases} y & 0 \le y < x \\ x + B & y \ge x \end{cases} \tag{3}$$

The **competitive ratio** $\mathrm{cr}(x, y)$ for a given pair of x and y is defined as the ratio between the costs of the online and offline algorithms:

$$\mathrm{cr}(x, y) = \frac{\mathrm{cost}_{\mathrm{online}}(x, y)}{\mathrm{cost}_{\mathrm{offline}}(y)} \tag{4}$$

The **expected competitive ratio**, denoted as CR, is defined as the ratio between the expected cost of an online algorithm and that of the offline algorithm [16]:

$$\mathrm{CR} = \frac{\mathbb{E}_y[\mathbb{E}_x[\mathrm{cost}_{\mathrm{online}}(x,y)]]}{\mathbb{E}_y[\mathrm{cost}_{\mathrm{offline}}(y)]} \quad (5)$$

Our objective is to select the strategy of idling time x such that the **worst case** CR ($\max_y \mathrm{CR}$) is minimized.

B. Existing Solutions

For SSV, one strategy commonly used in the design[1] is that the engine is turned off immediately when the car stops. This strategy (with the short name **TOI** for **T**urning **O**ff **I**mmediately) has a fixed cost of B for any stop length y. For vehicles without SSS, the drivers may be reluctant to turn off the engine because of the concerns on the engine wear or other needs. This strategy (with the short name **NEV** for **NEV**er) certainly incurs large cost when the stop time is long. In the following, we review existing online algorithms proposed in the context of the ski rental problem.

A **deterministic** online algorithm chooses a fixed x in (3). Karlin et al. [8] prove that among all possible deterministic algorithms, the strategy of $x = B$ gives the smallest worst case $\mathrm{cr}(x, y)$:

$$\min_x \max_y \mathrm{cr}(x, y) = \max_y \mathrm{cr}(B, y) = 2 \quad (6)$$

We use **DET** to denote this online algorithm.

If we consider the metric of the worst case CR, DET is not the best strategy. Karlin et al. [16] further propose a randomized online algorithm, which can guarantee that the worst case CR is no larger than $e/(e-1)$ for any distribution of y. This bound is also proven to be the smallest that any online algorithm can provide with no further statistical information on y. This algorithm, denoted as **N-Rand**, selects the idling time x based on the probability density function $p(x)$ as follows

$$p(x) = \begin{cases} \frac{1}{B(e-1)} e^{\frac{x}{B}} & 0 \le x \le B \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Recently, Khanafer et al. [9] propose to include the first-moment (the average) μ or second-moment of the stop length as additional statistical information. It then derives a revised randomized algorithm to minimize the largest $\widetilde{\mathrm{CR}}$, where

$$\widetilde{\mathrm{CR}} = \mathbb{E}_y\left[\frac{\mathbb{E}_x[\mathrm{cost}_{\mathrm{online}}(x,y)]}{\mathrm{cost}_{\mathrm{offline}}(y)}\right] = \mathbb{E}_y\left[\mathbb{E}_x\left[\frac{\mathrm{cost}_{\mathrm{online}}(x,y)}{\mathrm{cost}_{\mathrm{offline}}(y)}\right]\right] \quad (8)$$

The definition in (8) is the expectation of competitive ratios, while the definition (5) is the ratio of expected cost of online strategy over the expected cost of optimal offline strategy. From the driver's perspective, he/she would care much more about the expected cost as defined in (5), than the expected competitive ratio as defined in (8). In our problem setting, we assume that some statistics of stop length y is known. Under this assumption, the expected offline cost $\mathbb{E}_y[\mathrm{cost}_{\mathrm{offline}}(y)]$, or the denominator of (5), is constant, as shown in Equation (13). Hence, minimizing the competitive ratio as defined in (5) is equivalent to minimizing the out-of-pocket expense.

While we recognize that there might be some other applications where (8) is more suitable, we adopt the definition of CR in (5) as our minimization goal, as in most other works on the ski rental problem. The minimization of **worst case** CR is a typical objective in the online algorithm research community, for problems with no or limited knowledge about future information. If the full distribution of the stop lengths is known, a possibly better objective is to minimize the **average** CR, as in [17].

With the available information on μ, if $\mu \le \frac{2(e-2)}{e-1}B = 0.836B$, the probability density function of x is derived as in (9); otherwise, it is the same as **N-Rand**.

$$p(x) = \begin{cases} \frac{1}{B(e-2)}\left(e^{\frac{x}{B}} - 1\right) & 0 \le x \le B \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

The upper bound on CR' is proven to be $1 + \frac{\mu}{2B(e-2)}$. We denote this strategy as **MOM-Rand**.

C. Other Related Works

Besides the deterministic [8] and randomized [16] online algorithms for the classical ski rental problem, there are related works on exploiting additional statistical information to improve the performance of online algorithms. Fujiwara et al. [17] assume the distribution of input (like stop length y in anti-idling problem) is exponential or uniform, and then minimize the average case cost. Xu et al. [18] assume the input is geometrically distributed, and they use the average value of input to make the rent or buy decision. Dong et al. [19] propose to use different subsets to classify input to determine the average case cost for geometrical distribution. These works rely on the full distribution information which are usually difficult to get a priori. Khanafer et al. [9] propose an algorithm based on the knowledge of average input length or its second moment.

Except for statistical information, there are other works that use some prediction techniques to improve the performance. Lu et al. [20] introduce the use of look ahead window (the knowledge on whether or not the stop length y is within the window) to minimize competitive ratio bound against uncertainty. Hwang et al. [14] use average input length to predict the next one for system level power management. Ramanathan et al. [13] use the previous input to predict the next, however, the correlation of consecutive inputs is not obvious according to current driving data as discussed in Section V-C. Yuan et al. [15] calculate the processor utilization ratio according to the estimated remaining workload, and compare it with the effective cooling rate to determine whether to switch the processor into sleep mode.

Regarding vehicle idling, the idling costs of different vehicles are summarized in [21], and the estimations of restart cost are reported in [7] and [22]. Hill et al. [23] look at the emissions from idling, investigate into the matter of diesel particulate for school buses, and evaluate the effectiveness of different retrofit solutions like ultra-low sulfur diesel fuel and diesel particulate filter. Based on the observation that the engine may not be shut down as it is sometimes needed to provide power for heating etc., Stodolsky et. al. [24] propose

[1] see e.g., http://en.wikipedia.org/wiki/Start-stop_system

to use alternative power sources while the engine is off, including direct-fired heater, auxiliary power units, and truck stop electrification. These solutions are evaluated for transit buses in [25].

Xu et al. [26] implement an intervention system which prevents excessive idling for school buses, motivated by idling's environmental impact. A stop longer than 120 seconds is viewed as excessive idling, and the engine is shut down when that happens. Dupuis et al. [27] propose to automatically shut down the engine to prevent fines on excessive idling. Winslow [28] proposes to design an anti idling alarm function, which is designed to prevent fines on excessive idling. From the algorithm design point of view, these three works implicitly propose a form of deterministic online algorithm c-DET, where c is, for example, 120 seconds for Xu et al. [26]. However, as can be seen in the discussion in Section II-B, these algorithms are not able to guarantee the same competitive ratio as [8]. Compared to these works, our work is the first to formally formulate the vehicle idling decision problem as a ski-rental problem and apply the existing solutions on online algorithm design. Furthermore, we propose a constrained ski rental problem with two new effective statistics and design online algorithms with strong performance guarantees.

In the following, we look at additional statistical information of the stop length that can help provide better performance guarantees than the existing solutions when distribution is unknown. We use the definition of CR in (5) to give the competitive analysis, because of its direct relationship with the expected cost of the online algorithm.

III. Anti-idling with Effective Statistics

The classic ski rental problem aims at improving the CR when the future information is unknown a-priori. Accurate prediction is extremely difficult in many scenarios, however, effective statistical information can be available. In this section, we discuss the effective statistics that can be helpful to improve the online algorithms for the ski rental problem. We call them constrained ski rental problem, as the statistical information is introduced as additional constraints to the original one [9].

A. Effective Statistics

The first moment (mean) and second moment are commonly used statistics to characterize a random variable. However, they are usually affected by extreme values. For the ski rental problem, the length of an extremely long stop is not necessarily informative. This is evidential in the optimal offline strategy: once the stop length y is longer than B, to what extent its length exceeds B will not affect the optimal offline decision, and the engine should be turned off immediately. Similarly, the behavior of DET is independent from the actual length y if $y > B$: it only waits until time B to turn off the engine.

Furthermore, a given first moment cannot necessarily help find the best strategy. For example, for stops with mean value smaller than B ($B = 28$, for instance), turning off immediately (**TOI**) seems to be a preferred strategy. However, we can easily find a counterexample. Suppose there are 10 stops, 9 with 1 second, and 1 with 1000 seconds, the mean is about 100 seconds. On average, these stops are long. However, **DET** incurs a smaller total cost (=65) than **TOI** (=280).

We observe that the expected length is still meaningful when stops are shorter than B, and propose to use μ_{B^-} (the expected length of stops given they are short) as defined in (10). For the stops with length longer than B, we use their total probability q_{B^+}, which is defined in (11). Here $q(y)$ represents the distribution of the stop length y, which is unknown before making an online decision. B is the break-even value defined in (1).

$$\mu_{B^-} = \int_{0^+}^{B} y q(y) dy \tag{10}$$

$$q_{B^+} = 1 - \int_{0^+}^{B} q(y) dy \tag{11}$$

Now all the possible distributions of stop length y can be described by the set $\mathcal{Q}$

$$\mathcal{Q} = \{\, q(y) | q(y) \geq 0,\ (10) \text{ and } (11) \text{ are satisfied.}\} \tag{12}$$

With these two constraints, the expected cost of the offline algorithm is

$$\mathop{\mathbb{E}}_{y}[\text{cost}_{\text{offline}}(y)] = \mu_{B^-} + q_{B^+} B \tag{13}$$

and the expected cost of **DET** is

$$\mathop{\mathbb{E}}_{y}[\text{cost}_{\text{DET}}(y)] = \mathop{\mathbb{E}}_{y}[\text{cost}_{\text{online}}(B, y)] = \mu_{B^-} + 2 q_{B^+} B \tag{14}$$

both of which are constant for a given pair of μ_{B^-} and q_{B^+}. Also, an upper bound on the expected offline cost can be derived as $\mu_{B^-} + q_{B^+} B \leq B$. This is consistent with the intuition that no online algorithm (including TOI, whose expected cost is always B) can outperform the offline algorithm.

B. Problem Formulation

Our target is to find an online algorithm that defines the probability distribution $p(x)$ of the idling time x with the given information of μ_{B^-} and q_{B^+}, such that it provides the lowest upper bound on the CR (and consequently the expected online cost). The algorithm designer's strategy space $\mathcal{P}$ defines the set of all possible $p(x)$ available for the designer.

$$\mathcal{P} = \left\{ p(x) | p(x) \geq 0,\ \int_{0^+}^{+\infty} p(x) dx = 1 \right\} \tag{15}$$

For a given pair of μ_{B^-} and q_{B^+}, the adversary's strategy space is $\mathcal{Q}$ as defined in (12), and the offline cost (13) becomes constant. Minimizing the worst case CR in (5) is equivalent to solving the minimax problem defined in (16)

$$\min_{p \in \mathcal{P}} \max_{q \in \mathcal{Q}} \ J(p, q) \tag{16}$$

where $J(p, q)$ is the expected online cost with strategy $p(x)$ and stop length distribution $q(y)$

$$J(p, q) = \mathop{\mathbb{E}}_{y}[\mathop{\mathbb{E}}_{x}[\text{cost}_{\text{online}}(x, y)]] \tag{17}$$

We first consider the solution format. Similar to the randomized algorithm (N-Rand) [16], it can be proved that $\forall x > B$, $p(x) = 0$ (see Section III-C). In other words, the optimal online strategy only selects idling time x no larger than B.

Furthermore, we observe that N-Rand has a continuous probability density function (pdf) for $x \in [0, B]$. The deterministic online algorithm (DET) exhibits the same optimal behavior as the offline algorithm when the stop length y is less than B. On the other hand, the solution of TOI follows the online strategy when $y > B$. Both DET and TOI can be regarded as discrete probability distributions, represented with Dirac delta function. Thus, we propose a generic solution format for the designer's strategy $p(x)$, to include the discrete and continuous distributions simultaneously:

$$p(x) = h(x) + \sum_{i=0}^{N} \theta_i \delta(x - d_i) \tag{18}$$

where $h(x)$ is the continuous probability density function, and $\delta(x - d_i)$ is the Dirac delta function representing discrete distribution at d_i. In Equation (18), there are $N+1$ discrete distributions, represented by $\delta(x - d_i)$. d_i can be any value ranging from ε (an arbitrarily small positive number, representing the algorithm TOI) to B. These discrete distributions represent d_i-**DET** strategies, which keep the engine idle until d_i seconds and then turn it off. θ_i is the probability for the corresponding discrete distribution at d_i.

We now summarize the steps to solve the constrained ski rental problem in (16). In Section III-C, we determine the range of strategy space, and prove that $p(x > B) \equiv 0$. In Section III-D, we first construct the dual problem of the max problem in (16) (constrained by (10) and (11)), and convert the original minimax problem (16) to a minimization problem. Because of the difficulty to find a closed form optimal solution, we restrict the solution space by replacing one inequality constraint (38b) with its equality version, and solve the minimization problem. This will result in ONAIR, a closed-form but sub-optimal solution. In Section III-E, we try to derive the optimal solution by numerically solving the constrained ski rental problem (16). The closed-form sub-optimal solution and the optimal numerical solution are then compared in Section III-F.

C. Range of Strategy Space

In this section, we prove that the strategy space of the online algorithm is limited to $x \leq B$.

Theorem 1. *The optimal value to the optimization problem in (16) is the same as the one defined in (19)-(20), which has the same objective function but with a reduced feasible set.*

$$\min_{p \in \mathcal{P}'} \max_{q \in \mathcal{Q}} \quad J(p, q) \tag{19}$$

where $J(p, q)$ is defined in (17), $\mathcal{Q}$ defined in (12), B defined in (1), and

$$\mathcal{P}' = \left\{ p(x) \,\middle|\, p(x) \geq 0; \int_{0^+}^{B} p(x)dx = 1; \int_{B^+}^{+\infty} p(x)dx = 0 \right\} \tag{20}$$

Remark. *Theorem 1 means that, with the known statistics μ_{B^-} and q_{B^+} (defined in (10) and (11) respectively), reducing the feasible set from $\mathcal{P}$ to $\mathcal{P}'$ would not incur optimality loss of the optimization problem in (16).*

Proof. Suppose there exists an optimal strategy $p_1(x)$ in $\mathcal{P} \backslash \mathcal{P}'$, i.e., $\int_{B^+}^{+\infty} p_1(x)dx > 0$. For any such $p_1(x)$, we can construct another strategy $p_2(x) \in \mathcal{P}'$ as in (21).

$$p_2(x) = \begin{cases} p_1(x) & x < B \\ (\int_{B}^{+\infty} p_1(x)dx) \cdot \delta(x - B) & x = B \\ 0 & x > B \end{cases} \tag{21}$$

For any distribution $q(y) \in \mathcal{Q}$, we construct a new distribution $q'(y)$ of y as follows

$$q'(y) = \begin{cases} q(y) & y \leq B \text{ or } y > 2B \\ 0 & B < y < 2B \\ (\int_{B^+}^{2B} q(y)dy) \cdot \delta(y - 2B) & y = 2B \end{cases} \tag{22}$$

It is easy to see that $q'(y) \in \mathcal{Q}$ as well. In addition,

$$\int_{2B}^{+\infty} q'(y)dy = q_{B^+} \tag{23}$$

$$\forall x \leq B, \int_{x}^{+\infty} q'(y)dy = \int_{x}^{+\infty} q(y)dy \tag{24}$$

We now prove that $J(p_1(x), q'(y)) \geq J(p_2(x), q(y))$. Firstly, we compare $\mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'}$ and $\mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_q$

Case 1: if $x < B$,

$$\begin{aligned} & \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'} \\ = & \int_{0^+}^{x} yq'(y)dy + \int_{x^+}^{+\infty} (x + B)q'(y)dy \\ = & \int_{0^+}^{x} yq'(y)dy + (x + B) \int_{x^+}^{+\infty} q'(y)dy \\ = & \int_{0^+}^{x} yq(y)dy + (x + B) \int_{x^+}^{+\infty} q(y)dy \text{ (by Eqn (22) and (24))} \\ = & \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_q \end{aligned} \tag{25}$$

Case 2: if $x \in [B, 2B)$,

$$\begin{aligned} & \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'} \\ = & \int_{0^+}^{B} yq'(y)dy + \int_{B^+}^{x} yq'(y)dy \\ & + \int_{x^+}^{2B^-} (x + B)\, q'(y)dy + \int_{2B}^{+\infty} (x + B)\, q'(y)dy \\ \geq & \int_{0^+}^{B} yq'(y)dy + 0 + 0 + \int_{2B}^{+\infty} 2Bq'(y)dy \\ = & \mu_{B^-} + 2q_{B^+}B \text{ (by Equation (23))} \\ = & \mathbb{E}_y[\text{cost}_{\text{online}}(B, y)]_q \text{ (by Equation (14))} \end{aligned} \tag{26}$$

Case 3: if $x \geq 2B$,

$$\begin{aligned} & \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'} \\ = & \int_{0^+}^{B} yq'(y)dy + \int_{B^+}^{2B^-} yq'(y)dy + \int_{2B}^{x} yq'(y)dy \\ & + \int_{x^+}^{+\infty} (x + B)\, q'(y)dy \\ \geq & \int_{0^+}^{B} yq'(y)dy + 0 + \int_{2B}^{x} 2Bq'(y)dy + \int_{x^+}^{+\infty} 2Bq'(y)dy \\ = & \mu_{B^-} + 2q_{B^+}B \text{ (by Equation (23))} \\ = & \mathbb{E}_y[\text{cost}_{\text{online}}(B, y)]_q \text{ (by Equation (14))} \end{aligned} \tag{27}$$

Combining the above Equations (25)–(27) and the definition of $p_2(x)$ in (21), we have $\forall q(y) \in \mathcal{Q}$, $\exists q'(y) \in \mathcal{Q}$ such that

$$\begin{aligned} & J(p_1(x), q'(y)) \\ = & \int_{0^+}^{B^-} \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'}\, p_1(x)dx \\ & + \int_{B}^{2B^-} \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'}\, p_1(x)dx \\ & + \int_{2B}^{+\infty} \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q'}\, p_1(x)dx \\ \geq & \int_{0^+}^{B^-} \mathbb{E}_y[\text{cost}_{\text{online}}(x, y)]_{q}\, p_1(x)dx \\ & + \mathbb{E}_y[\text{cost}_{\text{online}}(B, y)]_q (\int_{B}^{2B^-} p_1(x)dx + \int_{2B}^{+\infty} p_1(x)dx) \\ = & J(p_2(x), q(y)) \end{aligned}$$

Since we can find a suitable q' for every q, this means that $\max_{q \in \mathcal{Q}} J(p_1(x), q(y)) \geq \max_{q \in \mathcal{Q}} J(p_2(x), q(y))$ in general. In other words, strategy p_2 incurs no larger cost than strategy p_1. We can therefore restrict our feasible set to $\mathcal{P}'$ without deteriorating our cost function.

$\square$

D. Sub-optimal Closed-form Solution

We denote the expected online cost for a given $y \leq B$ as

$$C(p(x), y) = \int_{0^+}^{y} (x+B)p(x)dx + y\int_{y}^{B} p(x)dx \tag{28}$$

and the one for $y > B$ as

$$C'(p(x), y) = \int_{0^+}^{B} (x+B)p(x)dx \tag{29}$$

The expected online cost $J(p,q)$ can be represented as

$$\begin{aligned} J(p,q) &= \int_{0^+}^{+\infty} \mathbb{E}_x[\text{cost}_{\text{online}}(x,y)]q(y)dy \\ &= \int_{0^+}^{B} C(p(x),y)q(y)dy + \int_{B}^{+\infty} C'(p(x),y)q(y)dy \end{aligned} \tag{30}$$

To solve the minimax problem in (16), we first consider the subproblem $\max_{q\in Q} J(p,q)$, and construct its dual problem. To take the constraints (10) and (11) into account, the Lagrangian associated with the max problem can be defined as

$$\begin{aligned} L(q,\lambda_1,\lambda_2) =& J(p,q) + \lambda_1\left(-\int_{0^+}^{B} q(y)dy + 1 - q_{B^+}\right) \\ &+ \lambda_2\left(-\int_{0^+}^{B} yq(y)dy + \mu_{B^-}\right) \end{aligned} \tag{31}$$

where λ_1 and λ_2 are the Lagrangian multipliers, both of which are unconstrained.

We now expand the two terms on the right hand side in (30) with the defined generic solution format in (18). We observe that for $d_0 = \varepsilon$ or $d_N = B$, the adversary is completely constrained by the statistics μ_{B^-} and q_{B^+}, as the expected cost is a constant (B for $d_0 = \varepsilon$, $\mu_{B^-} + 2q_{B^+}B$ for $d_N = B$) for any $q(y)$ that satisfies (10) and (11). Hence, we process the discrete distributions $\theta_\varepsilon\delta(x-\varepsilon)$ and $\theta_B\delta(x-B)$ separately. We define $\tilde{p}(x) = h(x) + \sum_{i=1}^{N-1}\theta_i\delta(x-d_i)$, and $\theta_\varepsilon = \theta_0$, $\theta_B = \theta_N$.

Now the first term on the right hand side in (30) is

$$\begin{aligned} &\int_{0^+}^{B} C(p(x),y)q(y)dy = \int_{0^+}^{B} C(\tilde{p}(x),y)q(y)dy \\ &+ \int_{0^+}^{B} C(\theta_\varepsilon\delta(x-\varepsilon)+\theta_B\delta(x-B),y)q(y)dy \\ &= \int_{0^+}^{B} C(\tilde{p}(x),y)q(y)dy + \theta_\varepsilon B(1-q_{B^+}) + \theta_B\mu_{B^-} \end{aligned} \tag{32}$$

The Lagrangian in (31) can be partitioned into two parts:

$$L(q,\lambda_1,\lambda_2) = Obj\,(\lambda_1,\lambda_2) + \int_{0+}^{B} Con\,(\lambda_1,\lambda_2)\cdot q\,(y)dy \tag{33}$$

where

$$\begin{aligned} Obj(\lambda_1,\lambda_2) &= q_{B^+}\cdot\int_{0+}^{B}(x+B)\,h(x)dx + \sum_{i=1}^{N-1}\theta_i\,(d_i+B) \\ &+ \theta_\varepsilon B + 2\theta_B q_{B^+}B + \theta_B\mu_{B^-} + \lambda_1\,(1-q_{B^+}) + \lambda_2\mu_{B^-} \end{aligned} \tag{34}$$

$$Con\,(\lambda_1,\lambda_2) = C(\tilde{p}(x),y) - \lambda_1 - \lambda_2 y \tag{35}$$

By optimization theory [29], the dual function is

$$g\,(\lambda_1,\lambda_2) = \sup_{q\in\{\,q(y)|q(y)\geq 0\}} L\,(q,\lambda_1,\lambda_2) \tag{36}$$

(36) is finite only if the coefficient of q is non-positive, since the probability function $q(y) \geq 0$:

$$g\,(\lambda_1,\lambda_2) = \begin{cases} Obj\,(\lambda_1,\lambda_2) & Con\,(\lambda_1,\lambda_2) \leq 0 \\ \infty & \text{otherwise} \end{cases} \tag{37}$$

Due to the linearity of $J(p,q)$ on q, strong duality holds [29]. Now, the original minimax problem in (16) is equivalent to the minimization problem in (38).

$$\min_{p,\lambda_1,\lambda_2} \; Obj\,(p,\lambda_1,\lambda_2) \tag{38a}$$

$$s.t. \quad \forall 0\leq y\leq B, C(\tilde{p}(x),y) - \lambda_1 - \lambda_2 y \leq 0 \tag{38b}$$

$$\int_{0^+}^{B}\tilde{p}(x)dx = 1-\theta_\varepsilon-\theta_B \tag{38c}$$

$$h(x)\geq 0 \tag{38d}$$

$$0\leq\theta_i\leq 1, i=0,1,..,N \tag{38e}$$

However, it is very difficult to find a closed-form solution for (38). In the remainder of this subsection, we replace the inequality (38b) with equality

$$\forall 0\leq y\leq B, C(\tilde{p}(x),y) - \lambda_1 - \lambda_2 y = 0 \tag{39}$$

This will allow us to derive a closed form solution, but it is suboptimal as the resulted feasibility region is restricted. In Section III-E, the exact solution will be solved numerically by discretizing the minimax problem, and in Section III-F, the closed-form suboptimal solution is compared with the optimal numerical solution in terms of performance and applicability.

Substituting $C(\tilde{p}(x),y)$ in (39) by (28), we can derive the equality constraint as in (40):

$$\int_{0+}^{y}(x+B)\tilde{p}(x)dx + y\int_{y}^{B}\tilde{p}(x)dx - \lambda_1 - \lambda_2 y = 0 \tag{40}$$

The equality constraint (40) is differentiated with respect to y, to derive the following ordinary differential equation (ODE):

$$B\tilde{p}\,(y) + \int_{y}^{B}\tilde{p}\,(x)\,dx - \lambda_2 = 0 \tag{41}$$

Without differentiating the probability density function, we can define $f(x)$ as the cumulative density function for the continuous distribution $h(x)$.

$$\begin{aligned} &B\tfrac{d}{dy}f\,(y) + f\,(B) - f\,(y) - \lambda_2 \\ &+B\sum_{i=1}^{N-1}\theta_i\delta\,(y-d_i) + \sum_{i=1}^{N-1}\theta_i\,(1-u\,(y-d_i)) = 0 \end{aligned} \tag{42}$$

$u(y-d_i)$ is the Heaviside step function, whose value is zero when $y < d_i$, 0.5 when $y = d_i$, and one when $y > d_i$.

Using Laplace Transformation and plugging the initial conditions, we can get

$$\begin{aligned} F(s) = \; & \tfrac{1}{s-\frac{1}{B}}(\lambda_2+\theta_\varepsilon+\theta_B-1) \\ & -\tfrac{1}{s}(\lambda_2+\theta_\varepsilon+\theta_B-1) - \sum_{i=1}^{N-1}\theta_i\tfrac{e^{-d_i s}}{s} \end{aligned} \tag{43}$$

TABLE I: Vertices of the Convex Polytope

$(\theta_\varepsilon, \theta_B)$	strategy	condition
(0, 0)	**N-Rand**	$K_\varepsilon \geq 0, K_B \geq 0$
(0, 1)	**DET**	$K_B \leq K_\varepsilon, K_\varepsilon \leq 0 \| \| K_B \leq 0$
(1, 0)	**TOI**	$K_\varepsilon \leq K_B, K_\varepsilon \leq 0 \| \| K_B \leq 0$

Then with the fact that $f(B) = 1 - \sum_{i=0}^{N} \theta_i$, we can get the cumulative density function $f(x)$ using inverse Laplace Transformation, as in (44).

$$f(x) = \frac{1-\theta_\varepsilon-\theta_B}{e-1} e^{\frac{x}{B}} - \frac{1-\theta_\varepsilon-\theta_B}{e-1} - \sum_{i=1}^{N-1} \theta_i u\left(x-d_i\right) \quad (44)$$

Differentiating the cumulative density function, we can get $h(x)$ as follows.

$$h(x) = C_0 e^{\frac{x}{B}} - \sum_{i=1}^{N-1} \theta_i \delta\left(x-d_i\right) \quad (45)$$

where the coefficient $C_0 = \frac{1-\theta_\varepsilon-\theta_B}{B(e-1)}$. It should be noted that $h(x) \geq 0$, while $\theta_i \delta\left(x-d_i\right)$ is infinity when $x = d_i$ if $\theta_i > 0$. Due to the non-negativity of $h(x)$, all θ_i except for θ_0 (θ_ε) and θ_N (θ_B) have to be zero. With the result, we can get the designer's strategy as in (46).

$$p(x) = h(x) + \theta_\varepsilon \delta(x-\varepsilon) + \theta_B \delta(x-B) \quad (46)$$

It should be noted in (46), all probabilities for the discrete distributions in the designer's strategy $p(x)$ are zero, except for $d_i = \varepsilon$ and $d_i = B$.

Substituting (45) into (35), we can get the Lagrange multipliers (as functions of θ_i, the probability corresponding to the discrete distribution at d_i).

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = \frac{e}{e-1}\left(1-\theta_\varepsilon-\theta_B\right) \end{cases} \quad (47)$$

Substituting (47) into (34), the objective $Obj(\lambda_1, \lambda_2)$ is now a function of θ_ε and θ_B, as in (48).

$$\min_{\theta_\varepsilon, \theta_B} \quad K_\varepsilon \theta_\varepsilon + K_B \theta_B + \left(q_{B^+} B + \mu_{B^-}\right) \frac{e}{e-1} \quad (48)$$

where K_ε and K_B are constants, defined as

$$\begin{aligned} K_\varepsilon &= -q_{B^+} B \tfrac{e}{e-1} + B - \tfrac{e}{e-1} \mu_{B^-} \\ K_B &= -q_{B^+} B \tfrac{e}{e-1} + 2 q_{B^+} B - \tfrac{1}{e-1} \mu_{B^-} \end{aligned} \quad (49)$$

We incorporate the constraints that $p(x)$ is a valid probability function

$$s.t. \quad \begin{cases} \theta_\varepsilon + \theta_B \leq 1 \\ 0 \leq \theta_\varepsilon \leq 1, 0 \leq \theta_B \leq 1 \\ 0 \leq \mu_{B^-} \leq B, 0 \leq q_{B^+} \leq 1 \end{cases} \quad (50)$$

The linear programming problem with the objective in (48) and constraints in (50) can be solved using standard techniques in linear programming. Simply speaking, the constraints in (50) limit that θ_ε and θ_B are all finite. By the fundamental theorem in linear programming, the solution space of this LP problem forms a convex polytope, and the optimal solution is obtained in one of the three vertices. The strategy and associated cost to each vertex is summarized in Table I:

Case 1: $q_{B^+} B + \mu_{B^-} \leq \frac{e-1}{e} B$ and $\mu_{B^-} \leq (e-2)\, q_{B^+} B$. In this case, the vertex $(\theta_\varepsilon, \theta_B) = (0,0)$ has the smallest solution. The resulting strategy is the same as N-Rand defined in Equation (7). Consequently, the worst case CR is $\frac{e}{e-1}$ (a constant independent from μ_{B^-} and q_{B^+}).

Case 2: $2 q_{B^+} B + \mu_{B^-} \leq B$ and $(e-2) q_{B^+} B \leq \mu_{B^-}$. In this case, the vertex $(\theta_\varepsilon, \theta_B) = (0,1)$ has the optimal solution. The resulting strategy is the same as the deterministic online algorithm. The worst case CR is

$$\mathrm{CR} = \mathrm{CR}_{\mathrm{DET}} = \frac{\mu_{B^-} + 2 q_{B^+} B}{\mu_{B^-} + q_{B^+} B} \quad (51)$$

Case 3: $B \leq 2 q_{B^+} B + \mu_{B^-}$ and $\frac{e-1}{e} B \leq q_{B^+} B + \mu_{B^-}$. In this case, the vertex $(\theta_\varepsilon, \theta_B) = (1,0)$ gives the best objective value. The resulting strategy is the same as TOI. The worst case CR is

$$\mathrm{CR} = \mathrm{CR}_{\mathrm{TOI}} = \frac{B}{\mu_{B^-} + q_{B^+} B} \quad (52)$$

E. Optimal Numerical Solution

The strategy derived in Section III-D is sub-optimal as the feasible region is restricted by replacing inequality constraint (38b) with its equality version. To deal with the inequality constraint (38b), in this section, we investigate into the optimal solution for the constrained ski rental problem by solving the problem numerically. We discretize the original minimax problem, and convert it to a minimization problem by constructing the dual problem of the original max problem, .

In the discrete format, the designer can select a strategy to keep the engine idle until $i-1$ seconds and then turn it off, while $i \in \{1, 2, 3, ..., B\}$. We name this strategy as i-DET. Also the designer can use mixed strategy by selecting 1-DET with probability p_1, selecting 2-DET with probability p_2, and so on. The designer's strategy can be defined as in (53).

$$\mathcal{P}_{\mathrm{Discrete}} = \left\{ \vec{p} = [p_1, p_2, ..., p_B] \middle| \, 0 \leq p_i \leq 1, \ \sum_{i=1}^{B} p_i = 1 \right\} \quad (53)$$

On the other hand, the adversary can choose to give the designer a stop with length j while $j \in \{1, 2, 3, ...\}$, or use mixed strategy by giving the stop with length j with probability q_j. The adversary's strategy can be defined as in (54).

$$\mathcal{Q}_{\mathrm{Discrete}} = \left\{ \begin{array}{l} \vec{q} = [q_1, q_2, ...] \, | \, 0 \leq q_j \leq 1, \ \sum\limits_{j=1}^{\infty} q_j = 1, \\ \sum\limits_{j=B}^{\infty} q_j = q_{B^+}, \ \sum\limits_{j=1}^{B-1} j q_j = \mu_{B^-} \end{array} \right\} \quad (54)$$

Then, we consider the cost A_{ij} incurred when the designer selects i-DET, and the adversary gives a stop with length j. This cost can be defined as in (55).

$$A_{ij} = \begin{cases} B + (i-1) & i \leq j \\ j & i > j \end{cases} \quad (55)$$

In Figure 1, an example of the constrained ski rental problem is used to explain the discretized strategy $\mathcal{P}_{\mathrm{Discrete}}$, $\mathcal{Q}_{\mathrm{Discrete}}$, and cost A_{ij}, where $B = 4$. Each row represents a pure strategy i-DET for the designer, and each column represents a pure strategy with stop length j for the adversary. As in Theorem 1, the designer would not choose any i-DET strategy which idles more than B seconds, so there are only B rows for the designer. Since the designer's strategy space

$$\sum_{i=1}^{\infty} p_i = 1 \begin{cases} p_1 \\ p_2 \\ p_3 \\ p_4 \end{cases} \begin{array}{c} \overbrace{\begin{array}{ccc} q_1 & q_2 & q_3 \end{array}}^{\sum_{j=1}^{4-1} j q_j = \mu_{B^-}} \; \overbrace{\begin{array}{ccc} q_4 & q_5 & q_6 \end{array}}^{\sum_{j=4}^{\infty} q_j = q_{B^+}} \cdots \\ \left[\begin{array}{ccc:ccc} 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 5 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \\ 1 & 2 & 3 & 7 & 7 & 7 \end{array} \cdots \right] \end{array}$$

Fig. 1: Discretized Constrained Ski Rental Problem

is limited to $\{1, 2, 3, 4\}$, all pure strategies for the adversary with stops no shorter than B incur the same cost. The column $\{p_1, p_2, p_3, p_4\}$ that to the left of the matrix represents the probability of the corresponding i-DET in the mixed strategy for the designer, and the row above the matrix represents the probability distribution of the adversary's mixed strategy. It should be noted, the probability distribution of the adversary's strategy is constrained by μ_{B^-} and q_{B^+}.

Now we try to use the discretized designer strategy $\vec{p}$, the adversary's strategy $\vec{q}$, and the cost matrix A to formulate the original minimax problem. The expected cost for the ski rental problem described in Figure 1 can be defined in a general format as in (56).

$$J(p_i, q_j) = \sum_{i=1}^{B} \sum_{j=1}^{B-1} p_i A_{ij} q_j + q_{B^+} \sum_{i=1}^{B} p_i A_{iB} \tag{56}$$

The minimax problem describing the constrained ski rental problem can be defined as in (57).

$$\begin{aligned} &\min_{\substack{p_1, p_2, \\ \dots, p_B}} \max_{\substack{q_1, q_2, \\ \dots, q_{B-1}}} J(p_i, q_j) \\ &s.t. \quad \begin{cases} \sum_{j=1}^{B-1} q_j = 1 - q_{B^+}, \sum_{j=1}^{B-1} j q_j = \mu_{B^-} \\ q_j \geq 0, 1 - q_{B^+} - q_j \geq 0, 1 \leq j \leq B-1 \\ \sum_{i=1}^{B} p_i = 1 \\ p_i \geq 0, p_i \leq 1, 1 \leq i \leq B \end{cases} \end{aligned} \tag{57}$$

To solve the minimax problem, we first find the dual problem of the original max problem. To this end, we construct the Lagrangian as defined in (58). λ_{0j}, λ_{1j}, ν_1, and ν_2 are Lagrangian multipliers, where λ_{0j} and λ_{1j} are non-negative.

$$\begin{aligned} &L(q_j, \nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = \sum_{i=1}^{B} \sum_{j=1}^{B-1} p_i A_{ij} q_j + q_{B^+} \sum_{i=1}^{B} p_i A_{iB} \\ &+\nu_1 \left(- \sum_{j=1}^{B-1} q_j + 1 - q_{B^+} \right) + \nu_2 \left(- \sum_{j=1}^{B-1} j q_j + \mu_{B^-} \right) \\ &+ \sum_{j=1}^{B-1} \lambda_{0j} q_j + \sum_{j=1}^{B-1} \lambda_{1j} (1 - q_{B^+} - q_j) \\ &= \underbrace{q_{B^+} \sum_{i=1}^{B} p_i A_{iB} + \nu_1 (1 - q_{B^+}) + \nu_2 \mu_{B^-} + \sum_{j=1}^{B-1} \lambda_{1j} (1 - q_{B^+})}_{Obj(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j})} \\ &+ \sum_{j=1}^{B-1} \underbrace{\left(\sum_{i=1}^{B} p_i A_{ij} - \nu_1 - \nu_2 j + \lambda_{0j} - \lambda_{1j} \right)}_{Con(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j})} q_j \end{aligned} \tag{58}$$

$g(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = \sup_{q_j} L(q_j, \nu_1, \nu_2, \lambda_{0j}, \lambda_{1j})$ is the dual function. Lagrangian is linear on q_j, and a linear function is only bounded when the coefficient of the independent variable is zero, so the dual function is only bounded when $Con(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = 0$, when dual function is equal to $Obj(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j})$.

$$\begin{aligned} &g(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = \sup_{q_j} L(q, \nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) \\ &= \begin{cases} Obj(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) & Con(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned} \tag{59}$$

Now the dual problem of the original max problem can be constructed as in (60).

$$\begin{aligned} &\min_{\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}} Obj(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) \\ &s.t. \quad \begin{cases} Con(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = 0, 1 \leq j \leq B-1 \\ \lambda_{1j} \geq 0, 1 \leq j \leq B-1 \\ \lambda_{0j} \geq 0, 1 \leq j \leq B-1 \end{cases} \end{aligned} \tag{60}$$

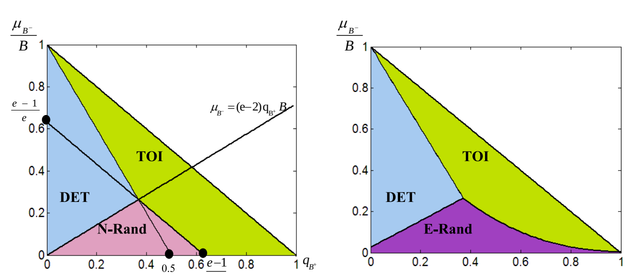
It should be noted that because Lagrangian (58) is linear on q_j, so strong duality holds. The maximum value of the original max problem is equal to the minimum value of the dual problem, so the original minimax problem is equivalent with the min problem as defined in (61).

$$\begin{aligned} &\min_{\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}, p_i} Obj(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) \\ &s.t. \quad \begin{cases} Con(\nu_1, \nu_2, \lambda_{0j}, \lambda_{1j}) = 0, 1 \leq j \leq B-1 \\ \lambda_{1j} \geq 0, 1 \leq j \leq B-1 \\ \lambda_{0j} \geq 0, 1 \leq j \leq B-1 \\ \sum_{i=1}^{B} p_i = 1, p_i \leq 1, -p_i \leq 0, 1 \leq i \leq B \end{cases} \end{aligned} \tag{61}$$

This is a linear programming problem, that can be solved with state-of-the-art solvers such as CPLEX [30]. It should be noted that the solution of p_i is dependent on μ_{B^-} and q_{B^+}, for a new pair of statistics, this linear programming problem has to be solved again. The strategy is denoted as **ONAIRE** (abbreviation for ONAIR Exact version).

F. Comparison of ONAIR and ONAIRE

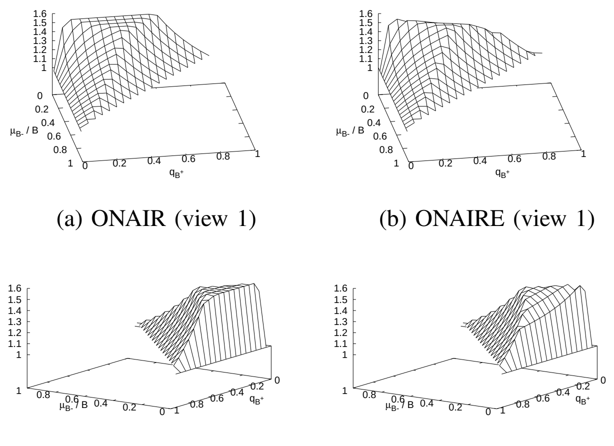
The rule to select TOI, DET, and N-Rand for ONAIR is illustrated in Figure 2(a). The selection is based on different values of the two statistics μ_{B^-} and q_{B^+}. In Figure 2(b), the rule to select different strategies for ONAIRE is exhibited. The μ_{B^-}, q_{B^+} plane is divided for different strategies. The area E-Rand (abbreviation for Exact Randomized algorithm) corresponds to the randomized algorithm selected by ONAIRE. Different from N-Rand, we cannot derive an analytic solution, and for different values of μ_{B^-} and q_{B^+}, the randomized strategy (represented by $\vec{p}$) is different.



(a) Rule Map of ONAIR (b) Rule Map of ONAIRE

Fig. 2: Strategy Rule Map Comparison

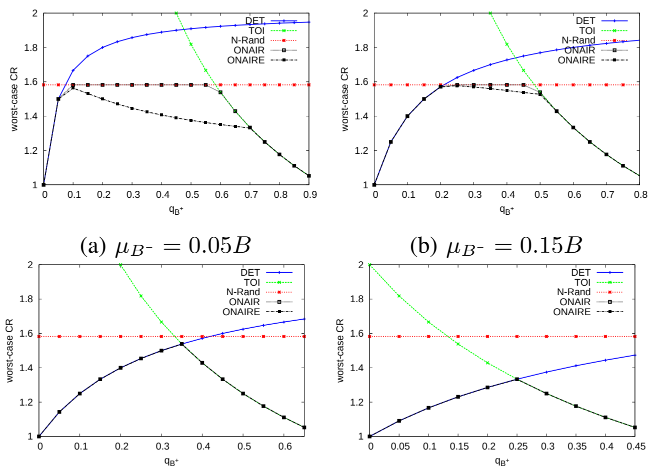
We now compare the strategy ONAIR with ONAIRE. The comparison is visualized in Figure 3. Figure 3 (a) and (c) exhibit two views of the worst case CR bound corresponding



(a) ONAIR (view 1) (b) ONAIRE (view 1)

(c) ONAIR (view 2) (d) ONAIRE (view 2)

Fig. 3: Worst Case CR Comparison



(a) $\mu_{B^-} = 0.05B$ (b) $\mu_{B^-} = 0.15B$

(c) $\mu_{B^-} = 0.3B$ (d) $\mu_{B^-} = 0.5B$

Fig. 4: Projected views

to ONAIR, while Figure 3 (b) and (d) show two views of the worst case CR bound corresponding to ONAIRE. For better illustration of the comparison, we also give projected views in Figure 4. As can be seen in the figures, in most cases, the sub-optimal solution performs as same as the optimal strategy. In some cases (when μ_{B^-} is relatively small), ONAIR is suboptimal, but provides close-to-optimal solutions. As can be seen later in Section V, ONAIR usually performs as well as ONAIRE for real driving data.

The implementations for the two strategies are different. ONAIR has a closed-form solution, and the selection of idling time x can be determined by evaluating some simple function. On the other hand, the optimal strategy only has a numerical solution. To get the optimal strategy, a complex linear programming would be solved for each μ_{B^-} and q_{B^+}, but this is impractical for embedded applications. An alternative approach is to solve the minimax problem for each pair of μ_{B^-} and q_{B^+} offline, and store all the strategies $p_i(\mu_{B^-}, q_{B^+})$ in a look-up table. However, this results in extra memory cost, especially because of the randomized strategy E-Rand in the purple area of Figure 2(b). For example, if the μ_{B^-},q_{B^+} plane is divided into a 1024×1024 grid, the time granularity is 1 second, the break-even value B is 28 seconds, and 4 bytes are used to store a value of probability, we need 56 MB memory to store all the coefficients for different cases of μ_{B^-}, q_{B^+}. In summary, ONAIRE performs better than ONAIR, at the cost of computation or memory cost.

The ONAIR and ONAIRE strategies can be implemented in vehicles with or without SSS. In current vehicles equipped with SSS, the engine is turned off immediately once vehicle stops. Historical stop lengths would be collected, which are used to update the two statistics value: μ_{B^-} and q_{B^+}. The idling or turning off decision is made according to the strategy, and executed automatically by SSS, without human in the control loop. In vehicles without SSS, the engine is turned on and off by the driver. The strategy can be implemented in car navigation system, which collects stop lengths information and updates statistics. A driving tip would be given to driver indicating when to turn off the engine.

ONAIR and ONAIRE rely on the estimation of break-even interval B, which is dependent on fuel price, battery price and ageing, starter life, etc. Among these factors, fuel price has a large variation, while the others are relatively stable within vehicle service life. To estimate break-even interval, the fuel price would be updated after refueling. These strategies also rely on the estimation of the statistics. To update the statistics μ_{B^-} and q_{B^+} in time, the latest stops can be used to update the current value of the statistics. In this way, the strategies can adapt to different communities or cities.

IV. Calculation of Break-even Interval B

In this section, we detail the break-even interval B for studying the tradeoff between idling and restart. We assume that the driver is interested in minimizing the total cost caused by idling reduction algorithm, including fuel consumption and amortized engine wear. However, we emphasize that the development of the online algorithms is independent from the actual value of B. US dollar is used as the default currency.

A. *Idling Cost*

Compared to restart, the cost of keeping the engine idle mainly comes from the extra consumption of fuel. In modern auto engines, the damage caused by excessive idling (spark plug fouling or lubricant contamination) is limited, compared with the cost of extra fuel [2].

The fuel cost during idling is dependent on the displacement of the engine. A quantized expression can be summarized as (62) [31], where $\mathrm{fuel}_{L/h}$ is the total fuel (in liter) consumed per hour, and D is the displacement of the engine.

$$\mathrm{fuel}_{L/h} = 0.3644 \times D + 0.5188 \tag{62}$$

Argonne National Laboratory [7] conducted an experiment on a 2011 Ford Fusion mid-sized sedan with a 2.5-L, 4-cylinder engine (175 HP) and 6-speed automatic transmission. The measured idling cost $\mathrm{fuel}_{cc/s}$ is about 0.279cc per second. The monetary cost of idling $\mathrm{cost}_{\mathrm{idling}\,/s}$ depends on the fuel price p_{gallon}, as in (63). If the fuel price is \$3.5 USD per gallon (1 gallon = 3785 cc), $\mathrm{cost}_{\mathrm{idling}\,/s}$ is about 0.0258 cent/s.

$$\mathrm{cost}_{\mathrm{idling}\,/s} = \mathrm{fuel}_{\mathrm{cc}\,/s} \times \frac{p_{\mathrm{gallon}}}{3785} \tag{63}$$

B. Cost of restart

For convenience, we normalize all the costs associated with restart with the cost of idling for 1 second ($\text{cost}_{\text{idling}\,/s}$).

1) Fuel: In terms of fuel consumption alone, fuel caused by one restart is estimated to be equivalent to the fuel cost during 10 seconds of idling. This estimation was reported by several experiments: Chrysler Canada in 1981 and European work in 1985 [2]; Natural Resources Canada's Office of Energy Efficiency on three 1999 model year vehicles [2]; and Argonne National Laboratory's test [7]. Based on these experiments, the fuel consumption of restart B_{fuel} can be safely calculated as 10 seconds of idling.

2) Engine wear: Engine wear is a critical concern from drivers, which may cause them to refuse to stop and restart the engine. Engine wear comes from possible damage to three components of the engine: the internal combustion engine (ICE) itself, starter, and starter battery.

$$\begin{aligned} B_{\text{engine},s} &= B_{\text{ICE},s} + B_{\text{starter},s} + B_{\text{battery},s} \\ B_{\text{engine},c} &= B_{\text{ICE},c} + B_{\text{starter},c} + B_{\text{battery},c} \end{aligned} \tag{64}$$

We distinguish two cases: SSV (with a second subscript 's' in the above equation) and the conventional vehicles without SSS (with a second subscript 'c').

a) ICE Wear: Despite the name "engine wear", ICE itself is the most durable among the three components. In modern stop-start systems, ICE is modified, so that ignition is adjusted according to the position of valves, in order to prevent further harm to ICE. Even without SSS, there is no evidence that restarting the engine causes significant wear to ICE, and we assume that this is negligible compared to the other costs. In other words, both $B_{\text{ICE},s}$ and $B_{\text{ICE},c}$ are estimated as zero.

b) Starter Wear: Compared with ICE, the starter is more vulnerable. In stop-start systems, starter is strengthened in order to deal with more frequent stop/start operations, while conventional vehicles may suffer from that. In the following, we discuss these two cases separately.

In SSV, the starter is usually strengthened. It is reported that SSS can allow a total of 1.2 million starts [32], typically enough for a car's lifetime. Due to the durability of SSV's starter, we estimate $B_{\text{starter},s}$ as 0.

For conventional vehicles, starter is much more vulnerable. We use the amortized replacement cost of the starter to estimate the cost per start. We refer to the relationship between starts per day and the vehicle service life as reported in [2]. The replacement cost of a starter ranges from \$55 to \$400, depending on many factors, e.g. the length of warranty, make, model, and engine size. Also, the labor cost of replacing the starter is significant, ranging from \$115 to \$225. An average cost per start $\text{cost}_{\text{starter},c}$ can be calculated by dividing the costs of replacement and labor by durability of the starter (between 20,000 and 40,000 starts/replacement). $\text{cost}_{\text{starter},c}$ is reported as 0.5 to 4 cents per start [2]. If the idling cost $\text{cost}_{\text{idling}\,/s}$ is 0.0258 cent/s, $B_{\text{starter},c}$ ranges from 19.38 to 155.04 seconds.

c) Starter Battery: The calculation of the restart cost associated to starter battery is more difficult to calculate, because of the uncertainty on the number of charging/discharging (called cyclic endurance) during a starter battery's lifetime.

TABLE II: Statistics of Stops in Different Areas

Location	Vehicles	Mean(μ)	Std(σ)	$P\{X \leq \mu + 2\sigma\}$
Atlanta	827	10.37	8.42	0.9091
Chicago	408	12.49	9.97	0.9534
California	291	9.37	7.68	0.9553
San Antonio	55	2.23	1.74	0.9636
Houston	59	4.85	3.10	0.9661
Aalborg (DK)	20	2.36	1.42	1.0000

Cyclic endurance depends on the depth of discharging and the pattern of charging/discharging cycles. For example, [2] reports that a battery with 1.75% depth of discharge could serve for 13250 cycles before failure. When the depth of discharge increases to 31%, the number of cycles decreases to 250. Batteries for stop-start systems are usually improved in order to meet the requirement of more frequent restarts. For example, VARTA stop-start pro batteries [33] could provide 3 times higher levels of cyclic endurance than conventional batteries, along with a very high deep discharge capability.

To estimate the cost of starter battery per start, we use the amortized battery cost by the possible number of stops during its warranty. For example, VARTA (an advanced stop-start battery) has a price of about \$230 (without labor cost) [34], with a warranty usually of 2-4 years. According to the driving data [35] [36], the total stops per day for different areas are listed in Table II. We consider the maximum of $\mu+2\sigma = 32.43$ as the estimated upper bound on the number of stops per day, such that more than 95% of the vehicles will fall in this range.

In the end, the lowest costs of starter battery for one start $\text{cost}_{\text{battery},s}$ and $\text{cost}_{\text{battery},c}$ are calculated to be 0.49 cents, and the $B_{\text{battery},s}$ and $B_{\text{battery},c}$ are at least 18 seconds.

3) Exhaust Emissions: The emission of CO_2 is proportional to the fuel consumed, thus one restart emits roughly the same amount of CO_2 as idling for 10 seconds. Evaluation on the monetary cost of emission largely depends on the legislation around the world. Carbon dioxide tax is introduced in many countries now. Similar to anti-idling rules, it varies a lot among different locations. Many developed countries have taxed the fuel directly for many years [37]. Without further information, we assume the cost incurred by CO_2 has already been included in the calculation of B_{fuel}.

Other emissions, including total hydrocarbons (THC), nitrogen oxides (NOx), and carbon monoxide (CO), are more relevant with the scrubber technology. The environmental concern on anti-idling is that these exhaust gas emissions from restarts is significantly larger than idling, due to the cooling of catalysts. According to the measurement by Argonne National Laboratory [7], restart causes emission of 44 mg THC, 6 mg NOx, and 1253 mg CO, while for every second of idling, emission of THC, NOx, and CO are 0.266 mg, 0.0097 mg, and 0.108 mg respectively.

However, most of the regulations against exhaust emissions have limited impact on excessive idling, as drivers are rarely charged due to exhaust emissions. Some countries have introduced regulations against NOx or CO, usually on manufacturers and power generation industry. Take the country of Sweden as an example, Nitrogen Oxidant is charged by about 4.3 Euros per kilogram of NOx (or the total emission of 166,667 restarts) [38]. Such a penalty is equal to \$0.0035 cents per restart, or the cost of an idling for 0.14 seconds. Hence, we assume that the impact of exhaust emissions is negligible in the calculation of break-even interval.

C. Summary of Break-Even Interval

For vehicles without SSS, the largest estimation variation for break-even interval comes from the starter, as discussed in Section IV-B. There are two reasons that we use the lowest value according to the break-even value's estimated range. First, with development of technology, the strength of starter is improving, so that the starter wear cost per start is decreasing. Second, $B_{\mathrm{starter},c}$ would decrease with higher fuel price. Hence, in the long run, $B_{\mathrm{starter},c}$ would decrease. For vehicles with SSS, the $B_{\mathrm{starter},s}$ = 0, and the estimation variation for break-even interval comes from $B_{\mathrm{battery},s}$. The reason we estimate starter battery wear cost using the warranty is that we assume the battery manufacturers determine the warranty according to the expected battery service life.

In summary, we estimate a minimum break-even interval of $B = 28$ seconds for SSV, and 47 seconds for conventional vehicles without SSS. We consider both the fuel consumption and mechanical wears. Hence, it addresses not only the environmental impact of vehicle idling reduction, but also car owners' concerns on damages to the car's starter and starter battery (the reasons why they are reluctant to shut down engines during idling).

V. Experimental Results and Discussion

In this section, we conduct experiments to evaluate the performance of the proposed online algorithm, including ONAIR which is a closed-form sub-optimal solution discussed in Section III-D, and ONAIRE, the optimal numerical solution as solved in Section III-E . We consider both SSV and the vehicles without stop-start systems. We use real-world driving data (Section V-A) as well as synthetic data (Section V-B) to study the CR of the proposed algorithms ONAIR and ONAIRE, and compare them with other strategies. In Section V-C, the correlation of consecutive stop lengths is investigated.

The solutions for comparison include TOI (Turning Off Immediately), NEV (Never turning off), DET (Deterministic Algorithm) [8], N-Rand (Randomized Online Algorithm) [16], and MOM-Rand [9]. For N-Rand, MOM-Rand algorithms, because these strategies rely on random numbers, while the time span of the data is usually short, so we repeat all the stop length data by 10 times to avoid significant statistical fluctuation. Also due to the limited time span of the data, the training data is not separated from the test data. The statistics μ_{B^-} and q_{B^+} are initialized using the entire dataset, because we assume the knowledge of the statistics.

Except for these existing algorithms, we also consider a clairvoyant algorithm (CLA for abbreviation), which has the knowledge of the full distribution of the stop length distribution. The full statistical information is usually difficult to get compared with several limited statistics as used in Section III, but this algorithm can serve as a benchmark to evaluate the effectiveness of the statistics used for improving online algorithm.

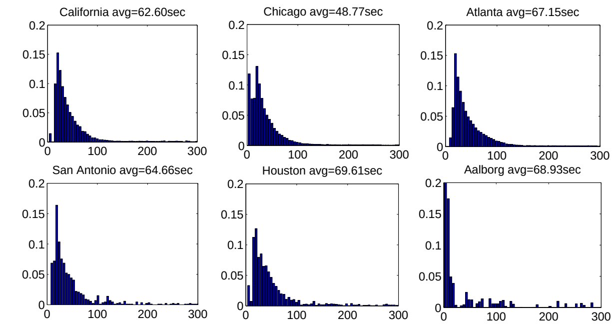


Fig. 5: Distribution of Stop Length

With the real-world driving data and the synthetic data, ONAIR and ONAIRE have very similar worst case or average case CR. The biggest difference is about 0.04 for the average CR for vehicles without SSS in the city of Aalborg. Hence, in the following, we focus our discussion on the comparison of ONAIR with other strategies.

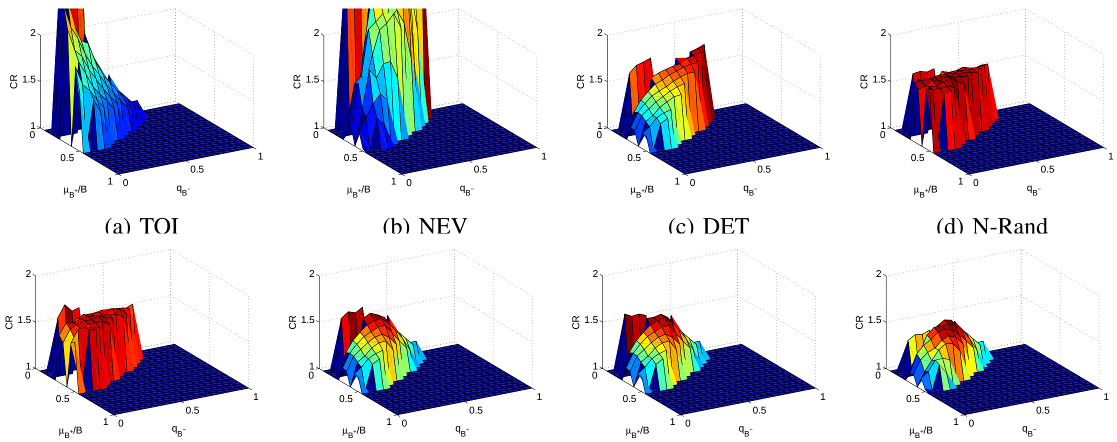
A. Real-World Driving Data

We first use real-world driving data to demonstrate the performance of our proposed control strategies and their advantage compared with current solutions. We select data released by the National Renewable Energy Laboratory (NREL) [35] in United States. These data are collected from: California, Chicago, Atlanta, San Antonio, and Houston. We also get another data trace from Aalborg University in Denmark [36]. For each vehicle, the driving data is recorded for one week. The statistical measurements of the data (by each area) are listed in Table II. Figure 5 depicts the probability distribution of the stop length for all the vehicles in these areas. These distributions are different from the exponential distribution (as assumed in [17]) according to the Kolmogorov-Smirnov test, mostly due to their heavy tails.

1) Worst Case CR*:* For SSV (where the break-even interval B is estimated at 28 seconds), the results are shown in Figure 6 for each of the six areas. For vehicles without SSS (where B is set to be 47 seconds), Figure 7 draws the comparison. From the figures, either ONAIR or ONAIRE always provides smaller worst case CR than the other strategies, except for CLA which has access to complete statistical information. Although ONAIR and ONAIRE are only aware of two statistics, they perform quite close to CLA.

Figure 6b shows the worst CR in different cities for SSV. Take the city Chicago as an example, although TOI is the optimal strategy for most vehicles in Chicago, there are some vehicles whose optimal strategy is N-Rand while TOI leads to a higher CR. Because ONAIR is able to select the best strategy among TOI, DET, and N-Rand with statistics μ_{B^-} and q_{B^+}, it also selects N-Rand for these vehicles. As a result, worst case $\mathrm{CR}s$ are similar between N-Rand and ONAIR for Chicago. The case in Aalborg has the same reason for the similarity of the worst case CR between N-Rand and ONAIR.

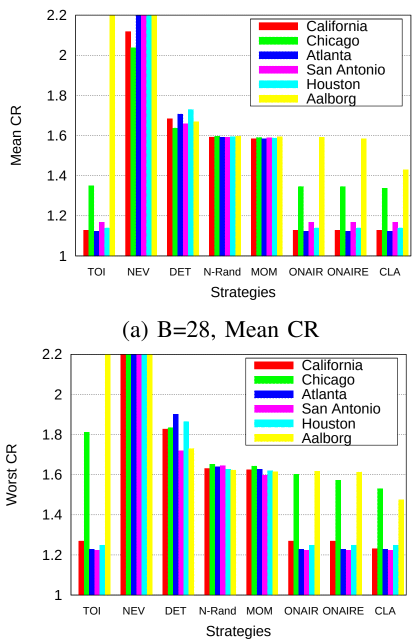
To give a more intuitive comparison, we divide the plane of μ_{B^-} and q_{B^+} into a 20×20 grid. half of the plane is infeasible region because of the constraint $\mu_{B^-} + q_{B^+}B \leq B$



(a) TOI (b) NEV (c) DET (d) N-Rand

(e) MOM-Rand (f) ONAIR (g) ONAIRE (h) CLA

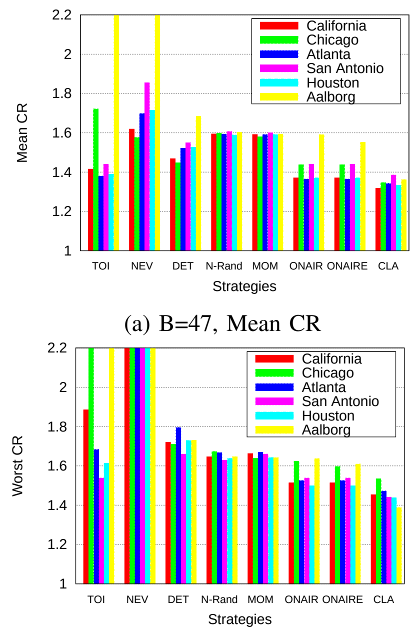
Fig. 8: Worst Case CR from Experimental Results, $B = 47$. (Note: CR $= 0$ represents no data in the figures)



(a) B=28, Mean CR

(b) B=28, Worst CR

Fig. 6: Result on real-world driving data: SSV



(a) B=47, Mean CR

(b) B=47, Worst CR

Fig. 7: Result on real-world driving data: w/o SSS

(offline expected cost $\mu_{B^-} + q_{B^+}B$ is smaller than that of any online algorithm, including TOI whose cost is B). Due to the limitation of the data, not every point is covered by the data used for evaluation, so the worst case CR bounds derived from experimental results are not as complete as the theoretical bounds in Figure 3. ONAIR and ONAIRE outperform other strategies in terms of lowering worst case CR bounds, except for the CLA strategy which has access to the complete distribution information.

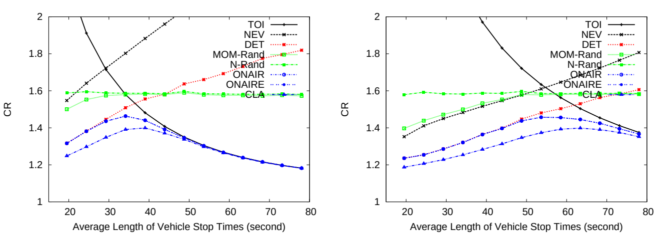
2) Average Case CR: In addition, we also compare the average case competitive ratio for different cities, as in Figure 6a and Figure 7a. While ONAIR and ONAIRE provide smaller bound of CR, they also outperform the other solutions in average CR. Among all the 1236 qualified vehicles (each vehicle experienced more than 30 stops), ONAIR achieves the best average CR in 1212 of them for SSV ($B = 28$). The mean CR of our algorithm is 1.13 (California), 1.35 (Chicago), 1.12 (Atlanta), 1.17 (San Antonio), 1.14 (Houston), and 1.58 (Aalborg, Denmark) respectively, lowest among all strategies. If $B = 47$ (for vehicles without SSS), our strategy achieves best performance in 1019 vehicles. The mean CR is 1.37 (California), 1.43 (Chicago), 1.36 (Atlanta), 1.43 (San Antonio), 1.37 (Houston), and 1.59 (Aalborg, Denmark) respectively, lower than the existing strategies. ONAIRE has similar performance with ONAIR. In summary, our algorithms not only provide the lowest upper bound on the CR, but also exhibit great performance in terms of the average CR in different areas.

Take the city of Aalborg in Figure 6a as an example, in all cities except for Aalborg, mean CRs of all vehicles are similar between TOI and ONAIR. This is because B is small for SSV, turning off immediately does not incur too much cost, when the stops are usually long, as seen in Figure 5. However, Aalborg has a different stop length distribution compared with others. Most stops in Aalborg are very short. TOI always turns the engine off immediately, so even a 1 or 2 second stop would incur the engine turning off and restart, which degrades its performance. On the other hand, ONAIR is able to select the best strategy among TOI, DET, and N-Rand. For Aalborg city, N-Rand is usually preferred, which performs better than TOI. That is the reason why the mean CR of TOI in Aalborg is much higher than that of ONAIR.

B. Synthetic Data

Finally, we generate synthetic data to validate the performance of the algorithms under different traffic conditions. Although different areas have different average stop length (possibly due to different traffic conditions), their shapes of the stop length distributions are quite similar, as in Figure 5. As an example, we generate simulation driving data by following the distribution of Chicago, but scaling its mean value. We then check the average case CR for each mean stop length.

Figures 9a and 9b illustrate the results. It can be seen that our strategies always achieve the lower CR under any traffic condition (average stop time). On the contrary, **DET** algorithm only functions well for good traffic conditions (with short average stop time), and **TOI** only works well for bad conditions (with long average stop time). The two randomized algorithms **N-Rand** and **MOM-Rand**, although robust, are consistently outperformed by our proposed algorithms. This validates our proposal that μ_{B^-} and q_{B^+} can provide valuable information to improve the online algorithm design.



(a) B = 28 (b) B = 47

Fig. 9: Average case CR vs. average stop lengths

As shown in Figure 9a and Figure 9b, when the average stop length of whole Chicago is large (most vehicles are expected to experience more long stops), ONAIR performs close to TOI in general. Because for long stops, the best strategy is to turn the engine off earlier, so that we can avoid excessive idling cost. When the average stop length of whole Chicago is small (most vehicles are expected to experience more short stops), ONAIR performs close to DET. Because for short stops, the best strategy is to keep the engine idle to avoid cost caused by frequent restart, where DET would perform well for most vehicles. The advantage area of TOI, DET, N-Rand can be seen from Figure 8. In both Figure 9a and Figure 9b, the curve representing ONAIR overlaps with that of ONAIRE. This means that, for anti-idling application, the sub-optimal solution based ONAIR strategy performs as well as the numerically optimal solution ONAIRE.

Compared with the CLA strategy which is aware of full statistical information, ONAIR and ONAIRE follows closely with CLA, so that they can also adapt to different driving environments. This is especially important for manufacturers who have no idea where the car would be driving. The advantage of CLA relies on the access to full statistical information that is usually not available. In addition, CLA would change with the adversary's distribution, while ONAIR and ONAIRE are valid for any distribution satisfying the constraints.

C. Dependence among Consecutive Stops

The worst case CR for the ski rental problem is the upper bound of the performance ratio between the online algorithm and the offline algorithm, as defined in (5). This bound is valid for stops with arbitrary distribution (under constraints if applicable), including the stops whose lengths are correlated.

For the vehicle idling problem, in some cases, there is certain correlation between consecutive stops, such as in heavy traffic jams. However, according to the data used in this work, there is no obvious correlation between previous stops' length and the next stop's length. Take the stops information from Chicago as an example, assuming that $B = 28$, hence the stops shorter than $B = 28$ are viewed as short stops, and the others as long stops. The probability of short stops $P\{y \leq B\} = 0.5487$. The conditional probability of short stops given the previous one is short is $P\{y_n \leq B | y_{n-1} \leq B\} = 0.5301$. The conditional probability of short stops given the previous two stops are short is $P\{y_n \leq B | y_{n-1} \leq B, y_{n-2} \leq B\} = 0.5114$. As a result, there is no significant dependence of next stop length on the previous ones. Nevertheless, this might be due to the source data we have at hand. We plan to investigate more on this issue in future work.

VI. CONCLUSIONS

In this paper, we formulate the vehicle idling reduction as the classical ski rental problem. Besides incorporating existing solutions, we propose a constrained ski rental problem with additional statistical information. We derive two online algorithms, the sub-optimal closed-form solution, and the optimal numerical solution. With real-world driving data and simulation, we demonstrate that the proposed algorithm is robust and advantageous for different types of vehicles under different traffic conditions.

REFERENCES

[1] US Environmental Protection Agency, "Low greenhouse gas emitting/eisa 141 compliant light duty vehicles model year," 2014. [Online]. Available: http://www.toronto.ca/transportation/onstreet/idling.htm

[2] GW Taylor Consulting, "Review of the incidence, energy use and costs of passenger vehicle idling," 2003. [Online]. Available: http://www.nrcan.gc.ca/

[3] Argonne National Labratory, "Reducing vehicle idling," 2013. [Online]. Available: http://www.transportation.anl.gov/engines/idling.html

[4] Toronto City Council, "Idling control by-law." [Online]. Available: http://www.toronto.ca/transportation/onstreet/idling.htm

[5] California Envrionmental Protection Agency, "California code of regulations title 13, div 3, ch 10." 2008. [Online]. Available: http://www.arb.ca.gov/

[6] Department for Transport, UK, "Road traffic (vehicle emissions) (fixed penalty) (england) regulations." [Online]. Available: http://www.legislation.gov.uk/uksi/2002/1808/contents/made

[7] Argonne National Labratory, "Which is greener: Idle, or stop and restart," 2013. [Online]. Available: http://www.afdc.energy.gov/uploads/publication/which_is_greener.pdf

[8] A. R. Karlin, M. S. Manasse, L. Rudolph, and D. D. Sleator, "Competitive snoopy caching," Algorithmica, vol. 3, no. 1-4, pp. 79–119, 1988.

[9] A. Khanafer, M. Kodialam, and K. Puttaswamy, "To rent or to buy in the presence of statistical information: The constrained ski-rental problem," Networking, IEEE/ACM Transactions on, vol. PP, no. 99, pp. 1–1, 2014.

[10] L. Lu, J. Tu, C.-K. Chau, M. Chen, and X. Lin, "Online energy generation scheduling for microgrids with intermittent energy sources and co-generation," in Proceedings of the ACM SIGMETRICS/international conference on Measurement and modeling of computer systems, 2013, pp. 53–66.

[11] R. El-Yaniv, R. Kaniel, and N. Linial, "Competitive optimal on-line leasing," Algorithmica, vol. 25, no. 1, pp. 116–140, 1999.

[12] T. Chantem, Y. Xiang, X. S. Hu, and R. P. Dick, "Enhancing multicore reliability through wear compensation in online assignment and scheduling," in Proceedings of the Conference on Design, Automation and Test in Europe, 2013, pp. 1373–1378.

[13] D. Ramanathan, S. Irani, and R. K. Gupta, "An analysis of system level power management algorithms and their effects on latency," Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on, vol. 21, no. 3, pp. 291–305, 2002.

[14] C.-H. Hwang and A. C.-H. Wu, "A predictive system shutdown method for energy saving of event-driven computation," ACM Transactions on Design Automation of Electronic Systems (TODAES), vol. 5, no. 2, pp. 226–241, 2000.

[15] L. Yuan, S. R. Leventhal, J. Gu, and G. Qu, "Talk: A temperature-aware leakage minimization technique for real-time systems," Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on, vol. 30, no. 10, pp. 1564–1568, 2011.

[16] A. R. Karlin, M. S. Manasse, L. A. McGeoch, and S. Owicki, "Competitive randomized algorithms for non-uniform problems," in Proceedings of the First Annual ACM-SIAM Symposium on Discrete Algorithms, 1990, pp. 301–309.

[17] H. Fujiwara and K. Iwama, "Average-case competitive analyses for ski-rental problems," Algorithmica, vol. 42, no. 1, pp. 95–107, 2005.

[18] Y. Xu and W. Xu, "Competitive algorithms for online leasing problem in probabilistic environments," in Advances in Neural Networks-ISNN 2004. Springer, 2004, pp. 725–730.

[19] Y. Dong, Y. Xu, and W. Xu, "The on-line rental problem with risk and probabilistic forecast," in Frontiers in Algorithmics. Springer, 2007, pp. 117–123.

[20] T. Lu, M. Chen, and L. L. Andrew, "Simple and effective dynamic provisioning for power-proportional data centers," IEEE Transactions on Parallel and Distributed Systems, vol. 24, no. 6, pp. 1161–1171, 2013.

[21] Argonne National Labrotory, "Idling reduction savings calculator." [Online]. Available: http://www.transportation.anl.gov/pdfs/idling_worksheet.pdf

[22] Environmental Protection Agency, "What you should know about truck engine idling." [Online]. Available: http://www.epa.gov/region1/eco/diesel/pdfs/Diesel_Factsheet_Truck_Idling.pdf

[23] L. B. Hill, N. J. Zimmerman, J. Gooch, and C. A. T. Force, A multi-city investigation of the effectiveness of retrofit emissions controls in reducing exposures to particulate matter in school buses. Clean Air Task Force Boston, MA, 2005.

[24] F. Stodolsky, L. Gaines, and A. Vyas, "Analysis of technology options to reduce the fuel consumption of idling trucks," Argonne National Lab., IL (US), Tech. Rep., 2000.

[25] E. Ziring and P. Sriraj, "Mitigating excessive idling of transit buses," Transportation Research Record: Journal of the Transportation Research Board, vol. 2143, no. 1, pp. 142–149, 2010.

[26] Y. Xu, V. Elango, R. Guensler, and S. Khoeini, "Idle monitoring, real-time intervention, and emission reductions from Cobb County, Georgia, school buses," Transportation Research Record: Journal of the Transportation Research Board, vol. 2340, no. 1, pp. 59–65, 2013.

[27] A. R. Dupuis and S. W. Callahan, "Anti idling control system and method of controlled engine shutdown," Feb. 19 2010, US Patent App. 12/708,981.

[28] D. B. Winslow, "Anti-idling alarm and method for vehicles," Apr. 7 2008, US Patent App. 12/098,673.

[29] S. Boyd and L. Vandenberghe, Convex optimization. Cambridge university press, 2009.

[30] CPLEX, IBM ILOG, "V12. 1: User's manual for cplex," International Business Machines Corporation, vol. 46, no. 53, p. 157, 2009.

[31] M. Barth, F. An, T. Younglove, C. Levine, G. Scora, M. Ross, and T. Wenzel, Development of a comprehensive modal emissions model. National Cooperative Highway Research Program, Transportation Research Board of the National Academies, 2000.

[32] Controlled Power Technologies, "Speedstart 1.2 million cranks." [Online]. Available: http://www.cpowert.com/SpeedStart-12-million-cranks

[33] Varta Coorporation. [Online]. Available: http://www.varta-automotive.com/

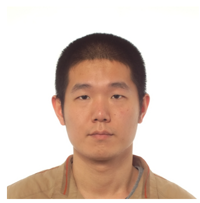
[34] Battery4Cars. [Online]. Available: http://www.battery4cars.co.uk/

[35] National Renewable Energy Laboarotory. [Online]. Available: http://www.nrel.gov/

[36] C. S. Jensen, H. Lahrmann, S. Pakalnis, and J. Runge, "The infati data," arXiv preprint cs/0410001, 2004.

[37] Wikipedia, "Carbon tax." [Online]. Available: http://en.wikipedia.org/wiki/Carbon_tax

[38] Swedish Environmental Protection Agency, "The swedish charge on nitrogen oxides cost-effective emission reduction." [Online]. Available: http://www.swedishepa.se/



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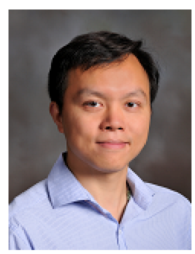

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