Competitive Online Optimization with Multiple Inventories: A Divide-and-Conquer Approach

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We study an online inventory trading problem where a user seeks to maximize the aggregate revenue of trading multiple inventories over a time horizon. The trading constraints and concave revenue functions are revealed sequentially in time, and the user needs to make irrevocable decisions. The problem has wide applications in various engineering domains. Existing works employ the primal-dual framework to design online algorithms with sub-optimal, albeit near-optimal, competitive ratios (CR). We exploit the problem structure to develop a new divide-and-conquer approach to solve the online multi-inventory problem by solving multiple calibrated single-inventory ones separately and combining their solutions. The approach achieves the optimal CR of $\ln\theta + 1$ if $N \leq \ln\theta + 1$, where N is the number of inventories and θ represents the revenue function uncertainty; it attains a CR of $1/[1 - e^{-1/(\ln\theta+1)}] \in [\ln\theta + 1, \ln\theta + 2)$ otherwise. The divide-and-conquer approach reveals novel structural insights for the problem, (partially) closes a gap in existing studies, and generalizes to broader settings. For example, it gives an algorithm with a CR within a constant factor to the lower bound for a generalized one-way trading problem with price elasticity with no previous results. When developing the above results, we also extend a recent CR-Pursuit algorithmic framework and introduce an online allocation problem with allowance augmentation, both of which can be of independent interest.

CCS Concepts: • **Mathematics of computing → Convex optimization**; • **Theory of computation → Design and analysis of algorithms**; **Online algorithms**; **Divide and conquer**; • **Applied computing → Decision analysis**.

Additional Key Words and Phrases: Inventory Constraints; Revenue Maximization; Resource Allocation; One-Way Trading.

ACM Reference Format:
Qiulin Lin, Yanfang Mo, Junyan Su, and Minghua Chen. 2022. Competitive Online Optimization with Multiple Inventories: A Divide-and-Conquer Approach. *Proc. ACM Meas. Anal. Comput. Syst.* 6, 2, Article 36 (June 2022), 28 pages. https://doi.org/10.1145/3530902

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1 INTRODUCTION

Competitive online optimization is a fundamental tool for decision-making with uncertainty. We have witnessed its wide applications spreading from EV charging [1–4], micro-grid operations [5, 6], energy storage scheduling [7, 8] to data center provisioning [9, 10], network optimization [11, 12], and beyond. Theoretically, there are multiple paradigms of general interest in the online optimization literature. Typical examples include the online covering and packing problem [13],

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2476-1249/2022/6-ART36 $15.00
https://doi.org/10.1145/3530902

online matching [14], online knapsack packing [15], one-way trading problem [16], and online optimization with switching cost [17, 18].

Among these applications and paradigms, we focus on optimizing the trading or allocation of limited resources, like inventories, cryptocurrency, budgets, or electric power, across a multi-round decision period with dynamic per-round revenues and allocation conditions. For example, in an online advertisement display platform (e.g., Google search), the operator allocates display slots to multiple advertisers (*cf.* inventories) with contracts on the maximum number (*cf.* capacity) of impressions [19]. At each round, the real-time search reveals the number of total available display slots and interesting slots for each advertiser (*cf.* allocation constraint). It also reveals the payoff of each advertiser from impressions at the display slots (*cf.* revenue function). The revenue of an advertiser relies on the number of obtained impressions and the quality of each impression relating to dynamic factors like the click rate and engagement rate [20].

The above observations motivate us to study the important paradigm – competitive online optimization problem under multiple inventories (OOIC-M), where a decision-maker with multiple inventories of fixed capacities seeks to maximize the per-round separable revenue function by optimizing the inventory allocation or trading at each round. The decision maker further faces two trading constraints at each round, the allowance constraint that limits the total trading amount of all inventories at the round and the rate constraints that limit the trading amount of each inventory at the round. The problem has two main challenges. First, as in online optimization under (single) inventory constraints [21, 22], the decision-maker does not have access to future revenue functions, while the limited capacity of each inventory coupling the online decisions regarding each inventory across time. Second, the allocation constraints couple the decisions across the multiple inventories at each round. The combination of allowance and rate limit constraints appears frequently and is with known challenges in online matching and allocation problems [14, 23–25].

In the literature, the authors in [22, 26] tackle the problem using the well-established online primal-and-dual framework [13, 27, 28]. They design a threshold function for each inventory with regard to the allocated amount, which can be viewed as the marginal cost of the inventory. They then greedily allocate the inventory at each round by maximizing the pseudo-revenue function defined by the difference between the revenue function and the threshold function. In contrast, in this paper, we propose a divide-and-conquer approach to online optimization with multiple inventories. Our approach is novel and provides additional insights to the problem. It allows us to separate the two challenges of the problem, 1) the online allocation for each inventory subject to the limited inventory capacity and unknown future revenue functions, and 2) the coupled allocation among multiple inventories due to the allowance constraint at each round. In the following, we summarize our contributions.

First, in Sec. 4, we generalize the CR-Pursuit(π) algorithm [21] to tackle the single inventory case, OOIC-S, which is an important component in our divide-and-conquer approach. We show that it achieves the optimal competitive ratio (CR) among all online algorithms for OOIC-S.

Second, in Sec. 5, we propose a divide-and-conquer approach to design online algorithms for online optimization under multiple inventories with dynamic revenue functions and trading constraints. We decompose the multiple inventory problem into several calibrated single inventory problems. We allocate the allowance among the subpoblems and combine their solutions. The approach achieves the optimal CR of $\ln\theta+1$ if $N \leq \ln\theta+1$, where N is the number of inventories and θ represents the revenue function uncertainty; it attains a CR of $1/[1-e^{-1/(\ln\theta+1)}] \in [\ln\theta+1, \ln\theta+2)$ otherwise, which is within a constant one to the lower bound.

Third, in Sec. 6, we discuss generalizations of our proposed approach to broader classes of revenue functions. We provide a sufficient condition for applying our online algorithm and derive

Table 1. Comparison of existing studies for online optimization under inventory constraints. Note that θ is a parameter representing revenue function uncertainty and N is the number of inventories.

Studies	Inventory		Revenue Function			Result and Technique	
	Single	Multi.	Linear	Concave	P.E.†	CR	Tech.
[16]	✓	✗	✓	✗	✗	$O(\ln\theta)$	Th.⋆
[21]	✓	✗	✓	✗	✓	$O(\ln\theta)$	CR-P⋆⋆
[26]	✓	✓	✓	✓	✗	$\leq \ln\theta + 2^{\ddagger}$	P&D⋆⋆⋆
[22]	✓	✓	✓	✓	✗	$\leq \ln\theta + 2^{\ddagger}$	P&D⋆⋆⋆
This paper	✓	✓	✓	✓	✓	$\ln\theta + 1^{\sharp}$, if $N \leq \ln\theta + 1$ $1/[1 - e^{-1/(\ln\theta+1)}]$, otherwise	Divide& Conquer

⋆ Threat-Based approach; ⋆⋆ CR-Pursuit; ⋆⋆⋆ The primal-and-dual framework.
† Concave revenue functions with price elasticity (See Sec. 6).
‡ The competitive ratio is sub-optimal when $N \leq \ln\theta + 1$, as shown in Sec. 5.5 and Table 3.
♯ Our CR is optimal if $N \leq \ln\theta + 1$ and is in $[\ln\theta + 1, \ln\theta + 2)$ otherwise (Theorem 12).

the corresponding CR it can achieve. For example, we consider revenue functions capturing the one-way trading problem with price elasticity, where only the results on the single inventory case are available in existing literature [21, 22]. We show that our approach obtains an online algorithm that achieves the optimal CR up to a constant factor.

Finally, our results in Sec. 5.2.2 generalize the online allocation maximization problem in [23] and the online allocation with free disposal problem in [19] by introducing allowance augmentation in online algorithms, which is of independent interest. We show that we can improve the CR from $e/(e-1)$ to $1/(1-e^{-1/\pi})/\pi$, when our online algorithms are endowed with π-time augmentation in allowance and allocation rate at each round.

2 RELATED WORK

We focus on the competitive online optimization problem with multiple inventories and dynamic allocation constraints. Our problem covers a couple of well-studied online problems, including the one-way trading problem [16, 29] where there is a single inventory with linear revenue functions, the online optimization under an inventory constraint [21, 22] where there is a single inventory, and the online fractional matching problem [23–25, 27, 30] where the revenue functions are linear functions with uniform scopes. Our results also reproduce the optimal CR under such settings. In these problems, multiple techniques are proposed, including Water-filling or BALANCING [24], threat-based approach [16], and CR-Pursuit framework [21]. Our online problem is also studied in [22], and a discrete counterpart is studied in [26]. Both studies are based on the online primal-and-dual framework. Compared with the existing study, we propose a novel divide-and-conquer approach. We show that our approach achieves a close-to-optimal CR, which notably matches the lower bound when the number of inventories is relatively small. Moreover, in Sec. 6, we apply our approach to different sets of revenue functions, which are not covered by the existing literature [22, 26]. We summarize the related literature on online optimization under inventory constraints in Table 1. We provide more detailed discussions in Sec. 3.3 and Sec. 5.4.

Our problem also covers a fractional version of the online ad display problem [14], which is an online matching problem with vertex capacity and edge value. No positive result is possible when the value is unbounded [19]. In [19], the authors consider a model of "free disposal," i.e., the online decision maker can remove the past allocated edges without any cost (but can not

Table 2. Notation Table.

Notation	Meaning
$g_{i,t}(\cdot)$	Revenue function of inventory i at slot t
$v_{i,t}$	Allocation of inventory i at slot t
C_i	Capacity of inventory i
A_t	Allowance of total allocation among all inventories at slot t
$\delta_{i,t}$	Maximum allocation of inventory i at slot t
θ	$\theta = p_{\max}/p_{\min}$, where $[p_{\min}, p_{\max}]$ is the range of the gradient of revenue functions
$\hat{a}_{i,t}$	Online allowance allocation to inventory i at slot t
$\hat{v}_{i,t}$	Online inventory allocation of inventory i at slot t
$\eta_{i,t}$	Online revenue of inventory i up to slot t
$\tilde{OPT}_{i,t}$	Optimal objective of OOIC-S$_i$ given allowance allocations and revenue functions up to slot t
OPT_t	Optimal offline total revenue of OOIC-M up to slot t
$\Phi(\pi)$	Maximum total online allocation of CR-Pursuit(π)

re-allocate the past edges). Here, we instead consider the case that the values of all edges are bounded in a pre-known positive range and no change on the past decision is available. We are interested in how the CR of an online algorithm behaves with regard to the uncertain range of value. Interestingly, by our divide-and-conquer approach, we can extend the results in the "free disposal" model to the irremovable setting. Also, we provide additional insight and results to the problem when considering an online augmentation scenario where the online decision maker is with a larger allowance and allocation rate at each round; see more details in Sec. 5.2.2.

Another related problem is the online knapsack packing problem [15]. In the problem, items are associated with weight and value and come online. An online decision maker with a capacity-limited knapsack determines whether to pack the item at its arrival to maximize the total value while guaranteeing that the total weight would not exceed the capacity. A single knapsack problem with infinitesimal assumption is studied in [15] with the application to key-work bidding. It can be viewed as a special case of the one-way trading problem [22], which is covered by the online optimization problem with a single inventory [21, 22]. The fractional multiple knapsack packing problem with unit weight is studied in [22], where the decision maker can pack any fraction of the item instead of a 0/1 decision. Our problem is also related to the online packing problem [13, 31] where the authors consider general packing constraints. Here, we focus on specific inventory constraints and allocation constraints.

3 PROBLEM FORMULATION

In this section, we formulate the optimal allocation problem with multiple inventories. We discuss the practical online scenario and the performance metric for online algorithms. We further discuss the state-of-the-art of the online problem. We summarize the important notions in Table 2.

3.1 Problem Formulation

We consider N inventories and a decision period with T slots. We denote the capacity of inventory i as C_i. At each slot $t \in [T]$, each inventory i is associated with a revenue function $g_{i,t}(v_{i,t})$, which represents the revenue of allocating or trading an amount of $v_{i,t}$ inventory i at slot t. However, at each slot $t \in [T]$, we are restricted to allocate or trade at most $\delta_{i,t}$ of inventory i, and the total allocation of all inventories at each slot t is bounded by the allowance A_t. Our goal is to find an optimal allocation scheme that maximizes the total revenue in the decision period while satisfying the allocation restrictions.

Overall, we consider the following problem,

$$\text{OOIC-M}: \max \sum_{i\in[N]} \sum_{t\in[T]} g_{i,t}(v_{i,t}) \tag{1}$$

$$\text{s.t.} \sum_{t} v_{i,t} \le C_i, \forall i \in [N], \tag{2}$$

$$\sum_{i} v_{i,t} \le A_t, \forall t \in [T], \tag{3}$$

$$0 \le v_{i,t} \le \delta_{i,t}, \forall t \in [T], i \in [N], \tag{4}$$

In OOIC-M, we optimize the inventory allocation $\{v_{i,t}\}_{i\in[N],t\in[T]}$ to achieve the maximum total revenue subjecting to the capacity constraint of each inventory (2), the allowance constraint at each slot (3), and the rate limit constraint for each inventory at each slot (4). Without loss of generality, we assume that $\delta_{i,t} \le A_t, \forall i, t$. We consider the following set of revenue functions, denoted as $\mathcal{G}$,

- $g_{i,t}(\cdot)$ is concave and differentiable with $g(0) = 0$;
- $g'_{i,t}(v_{i,t}) \in [p_{\min}, p_{\max}], \forall v_{i,t} \in [0, \delta_{i,t}]$.

We consider that $p_{\max} \ge p_{\min} > 0$ and denote $\theta = p_{\max}/p_{\min}$. The revenue functions capture the case where the marginal revenue of trading more inventory is non-increasing in the allocation amount but always between $p_{\min}$ and $p_{\max}$. We also discuss applying our approach to different sets of revenue functions with corresponding applications in Sec 6.

In the offline setting, the problem input is known in advance, and OOIC-M is a convex optimization problem with efficient optimal algorithms. However, in practice, we face an online setting, as we will describe next.

3.2 Online Scenario and Performance Metric

In the online setting, we consider that the pre-known problem parameters include the class of revenue function $\mathcal{G}$ and corresponding range $[p_{\min}, p_{\max}]$, the number of inventories N, and the capacity of each inventory $\{C_i\}_{i\in[N]}$. Other problem parameters are revealed sequentially. More specifically, at each slot t, the online decision maker without the information of the decision period T is fed the revenue functions $\{g_{i,v}(\cdot)\}_{i\in[N]}$, the allowance A_t and the allocation limits $\{\delta_{i,t}\}_{i\in[N]}$. We need to irrevocably determine the allocation at slot t, i.e., $\{v_{i,t}\}_{i\in[N]}$. After that, if the decision period ends, we stop and know the information of T. Otherwise, we move to the next slot and continue the allocation. We denote a possible input as

$$\sigma = \left(T, \{g_{i,t}(\cdot)\}_{i\in[N],t\in[T]}, \{A_t\}_{t\in[T]}, \{\delta_{i,t}\}_{i\in[N],t\in[T]}\right) \tag{5}$$

We use the *Competitive Ratio* as a performance metric for online algorithms. The CR of an algorithm $\mathcal{A}$ is defined as

$$\mathcal{CR}(\mathcal{A}) = \sup_{\sigma\in\Sigma} \frac{OPT(\sigma)}{ALG(\sigma)}, \tag{6}$$

where σ denotes an input, $OPT(\sigma)$ and $ALG(\sigma)$ denote the offline optimal objective and the online objective applying $\mathcal{A}$ under input σ, respectively. We use Σ to represent all possible input we are interested in. Specifically,

$$\Sigma \triangleq \{\sigma \,\big|\, T \in \mathbb{Z}^+, g_{i,t}(\cdot) \in \mathcal{G}, A_t \ge 0, \delta_{i,t} \ge 0, \forall i \in [N], t \in [T]\}, \tag{7}$$

In competitive analysis, we focus on the worst-case guarantee of an online algorithm, which is defined by the maximum performance ratio between the offline optimal and the online objective of the algorithm. In the online setting, we are facing two main challenges, 1) the decision maker does not know the future revenue functions, while the allocation now would affect the future decision due to the capacity constraint [21]; and 2) the online allowance constraints and rate constraints couple the decisions across the inventories, which are with known challenges in online matching and allocation problems [14, 24].

3.3 State of the Art

The online problem of OOIC-M has been studied in [22] under the same revenue function set $\mathcal{G}$. Given the gradients of function in $\mathcal{G}$ are bounded, we can view the problem as allocating items with infinitesimal sizes, and each item has a value between $p_{\min}$ and $p_{\max}$. Then, problem OOIC-M under revenue functions $\mathcal{G}$ can also be viewed as a continuous counterpart of [26]. When there is only one inventory ($N = 1$), the online problem reduces to the online optimization problem under inventory constraints [21, 22]. If we further restrict the revenue functions to be linear, it becomes the one-way trading problem studied in [16, 29]. Besides, when $p_{\max} = p_{\min}$, the online problem reduces to maximizing the total amount of allocation, which has been widely studied in [23–25, 27, 30]. Here, we introduce a novel divide-and-conquer approach to the problem and show that our approach can achieve a close-to-optimal CR to online OOIC-M. It also recovers the optimal CR for the above special cases covered by the problem. We provide a detailed discussion in Sec. 5.4. In Sec. 6, we show that our approach can be applied to different sets of revenue functions, which have not been studied in the existing literature.

Before proceeding, we discuss the two most relevant works in the literature, namely [26] and [22]. They both apply the online primal-dual analysis and design threshold functions for the online decision-making of OOIC-M. While the work [26] studied a discrete setting differing from the continuous setting studied in [22] and our work, it is known in [28] that the same threshold-based function can be directly applied to the continuous setting, attaining the same CR. In the following, let us reproduce the algorithm for the continuous setting and the CR achieved by [26]. The CR is better than the one proposed in [22], and thus we deem it state-of-the-art in the literature. We also compare the CR they achieve and ours in Sec. 5; see an illustration example in Fig. 2.

Let $\phi_i(w)$ denote the threshold function for each inventory i, where w refers to the amount of allocated capacity of the inventory and $\phi_i(w)$ can be viewed as a pseudo-cost of the allocation. At each slot, the algorithm determines the allocated amount $v_{i,t}$ of inventory i at slot t by maximizing the per-round pseudo-revenue, which is the difference between the revenue and the threshold function, i.e.,

$$\text{(P\&D):}\ \max \quad \sum_{i\in[N]}\left(g_{i,t}(v_{i,t}) - \int_{w_{i,t-1}}^{w_{i,t-1}+v_{i,t}} \phi_i(w)dw\right) \tag{8}$$

$$\text{s.t.} \quad \sum_{i} v_{i,t} \le A_t, \tag{9}$$

$$0 \le v_{i,t} \le \delta_{i,t}, \forall i \in [N], \tag{10}$$

where $w_{i,t-1}$ is the total online allocation of inventory i from the first slot to slot $t-1$. The algorithm is proposed in [22], which can be viewed as a continuous reinterpretation of the discrete algorithm in [26]. According to Appendix E of [26], we can apply the following threshold function given that the gradient of $g_{i,t}(v_{i,t})$ is uniformly bounded in the range $[p_{\min}, p_{\max}]$,

$$\phi_i(w) = \begin{cases} p_{\min} \cdot \frac{e^{w/C_i}-1}{e^{\chi}-1}, & w \in [0, \chi \cdot C_i]; \\ (p_{\min})^{\frac{1-w/C_i}{1-\chi}} (p_{\max})^{\frac{w/C_i-\chi}{1-\chi}}, & w \in [\chi \cdot C_i, C_i]; \end{cases} \tag{11}$$

where $\chi = W(\ln(\theta) \cdot e^{\ln(\theta)-1}) - \ln\theta + 1$ ($W(\cdot)$ is the Lambert-W function).

Proposition 1 ([26]). *With the threshold function (11), the threshold-based algorithm can achieve a competitive ratio of*

$$\tilde{\chi} = \frac{1}{1-e^{-\chi}}. \tag{12}$$

We provide the proof in Appendix A. We note that the proofs in [22] and [26] follow similar ideas based on the online primal-dual framework but are different in presentations as one is discussing in the continuous setting [22] and the other in the discrete setting [26]. As we are considering the continuous setting, our proof follows the same presentation discussed in [22] and applies the properties of the threshold function (11) discussed in [26].

4 CR-PURSUIT FOR SINGLE INVENTORY PROBLEM

In this section, we first discuss the problem with a single inventory. We extend the CR-Pursuit in [21] to cover the rate limit constraint and provide additional insights that will facilitate our algorithm design under the multiple inventories case.

In the single inventory case, OOIC-M is reduced to the following problem,

$$\text{OOIC-S}: \ \max \sum_t g_t(v_t) \tag{13}$$

$$\text{s.t.} \ \sum_t v_t \le C \tag{14}$$

$$\text{var.} \ 0 \le v_t \le \delta_t, \forall t, \tag{15}$$

where C denotes the capacity of the inventory, $g_t(v_t)$ represents the revenue of allocating v_t quantity of inventory at slot t, and δ_t is the rate limit restricting the maximum allocation at slot t. The goal is still to maximize the total revenue by determining the inventory allocation v_t at each slot. We focus on the online setting described in Sec. 3.2 with N specified to be one. We note that the OOIC-S has been studied in [22]. Also, the case under different assumptions on revenue functions and without rate limit has been studied in [21].

Here, we generalize the results in [21] to consider revenue function set $\mathcal{G}$ and involve rate limit constraint. The online algorithm CR-Pursuit(π), proposed in [21], is a single-parametric online algorithm with π as the parameter. At slot t, the algorithm first computes the optimal value of OOIC-S given the input revenue functions and rate limits up to t, which we denote as $OPT_S(t)$. It then determines the allocation $\hat{v}_t$ at slot t such that

$$g_t(\hat{v}_t) = \frac{1}{\pi} \left(OPT_S(t) - OPT_S(t-1)\right). \tag{16}$$

Under CR-Pursuit(π), we define the maximum total allocation of the algorithm,

$$\Phi(\pi) \ \triangleq \ \sup_{\sigma\in\Sigma} \sum_t \hat{v}_t, \tag{17}$$

where $\hat{v}_t$ is determined by (16). By design, we clearly have the following properties of the CR-Pursuit(π) algorithm.

Lemma 2. *We have* $\hat{v}_t \le \frac{1}{\pi} \cdot \delta_t$.

Proof. As $g_t(v_t)$ is increasing and concave function, and

$$g_t(\hat{v}_t) = \frac{1}{\pi}\left(OPT_S(t) - OPT_S(t-1)\right) \leq \frac{1}{\pi} \cdot g_t(\delta_t), \tag{18}$$

we have $\hat{v}_t \leq \frac{1}{\pi} \cdot \delta_t$.

Lemma 3 shows an upper bound on the online allocation, which guarantees the existence of $\hat{v}_t$ at each slot. It will also be useful for our algorithm design and CR analysis for OOIC-M, which we will discuss in Sec. 5.

Lemma 3. *CR-Pursuit(π)is feasible and π-competitive for OOIC-S if $\Phi(\pi) \leq C$.*

Proof. Considering an arbitrary input σ, we first note that it is clear that $\hat{v}_t \leq \delta_t$ according to Lemma 2. Then if $\Phi(\pi) \leq C$, we have $\sum_t \hat{v}_t \leq C$, i.e., it satisfies the inventory constraint under input σ.

Summarizing (16) over all t, we have that the online objective

$$ALG(\sigma) = \sum_{t=1}^{T} g_t(\hat{v}_t) = \frac{1}{\pi} OPT_S(T) = \frac{1}{\pi} \cdot OPT_S(\sigma), \tag{19}$$

where $OPT_S(\sigma)$ is the optimal offline objective. Thus the algorithm is π-competitive.

Lemma 3 shows that we can rely on characterizing $\Phi(\pi)$ to optimize the choice of π in CR-Pursuit(π). Further, we can interpret $\Phi(\pi)$ as the inventory the online algorithm CR-Pursuit(π) needs to maintain π-competitive. For example, suppose for an online algorithm, we now can utilize $\Phi(1)$ capacity of the inventory while the capacity of the offline optimal remains C. Then we can run CR-Pursuit(1) and achieve that same performance as the offline optimal, i.e., 1-competitive.

We have the following results on the upper bound on $\Phi(\pi)$.

Lemma 4. *We have*

$$\Phi(\pi) \leq \frac{\ln\theta + 1}{\pi} \cdot C. \tag{20}$$

We summarize the proof idea of Lemma 4 here while leaving the detailed proof in Appendix B. We first notice that a more general result in [21] can be extended to the case with rate limit constraint, as shown in Proposition 15 in Appendix B. Although the results in [21] do not cover the revenue functions we consider here, it covers the revenue functions in the maximizer of $\Phi(\pi)$ as special cases. This observation leads to Lemma 4.

According to the above discussion, we can provide the competitive analysis of CR-Pursuit(π) for OOIC-S. In particular, we set π as $\ln\theta + 1$.

Theorem 5. *For OOIC-S, CR-Pursuit($\ln\theta + 1$) is $\ln\theta + 1$-competitive. And it is optimal among all online algorithms for the problem.*

According to Lemma 3 and Lemma 4, it is clear that CR-Pursuit($\ln\theta + 1$) is feasible and $(\ln\theta + 1)$-competitive. Further, according to the results in [21, 22], we know that $\ln\theta + 1$ is the lower bound or the optimal CR to OOIC-S. Thus, CR-Pursuit($\ln\theta + 1$) is also optimal.

5 ONLINE ALGORITHMS FOR MULTIPLE INVENTORY PROBLEM

In this section, we introduce our divide-and-conquer online algorithm A&P(π) for OOIC-M, where π is a parameter to be specified. We first outline the algorithm structure. Following the structure, we then propose our general online algorithm for arbitrary N. We next show a simple and optimal online algorithm when N is relatively small. Finally, we summarize our algorithm and provide the competitive analysis. An illustration of our approach and results is shown in Fig 1.

5.1 Algorithm structure

We consider a divide-and-conquer approach for deriving online algorithms for OOIC-M. The general idea is that we can optimize OOIC-M by first allocating the allowance at each slot to the inventories and then separately optimizing the allocation of each inventory given the allocated allowance. More specifically, we define the following subproblem for each $i \in [N]$,

$$\text{OOIC-S}_i: \ \max \sum_t \tilde{g}_{i,t}(v_{i,t}) \tag{21}$$

$$s.t. \sum_t v_{i,t} \le C_i \tag{22}$$

$$0 \le v_{i,t} \le a_{i,t}, \forall t, \tag{23}$$

where $a_{i,t}$ is the allocated allowance to user i at slot t. $\tilde{g}_{i,t}(v_{i,t})$ is another algorithmic design space that allows us to exploit the online augmentation scenario when allocating the allowance allocation in Sec. 5.2.2. Under the offline setting, we note that such a decomposition is of no optimality loss. For example, we can choose $\tilde{g}_{i,t}(v_{i,t}) = g_{i,t}(v_{i,t})$ and set the allowance allocation $a_{i,t} = v^*_{i,t}$ for all i and t, where $\{v^*_{i,t}\}_{i\in[N],t\in[T]}$ is the offline optimal solution of OOIC-M. Then, optimizing the subproblems separately given the allowances would reproduce the offline optimal solution.

Following this structure, we can design an online algorithm that mainly consists of two steps at each slot t,

(1) Step-I: Determine the allowance allocation, $\{\hat{a}_{i,t}\}_{i\in[N]}$, irrevocably.
(2) Step-II: Determine the inventory allocation for each online OOIC-S$_i$, $\{\hat{v}_{i,t}\}_{i\in[N]}$, irrevocably.

We note that this divide-and-conquer approach allows us to separately tackle the two main challenges of the problem. First, the revenue functions come online while the allocation across the decision period is coupled due to the capacity constraint for each inventory, which is mainly handled by Step-II. Second, the online allowance constraints and the rate constraints couple the decisions across the inventories, which we tackle in Step-I. We can directly apply $\{\hat{v}_{i,t}\}_{t\in[T]}$ as the output of an online algorithm for each inventory i. We can view Step-II as solving OOIC-S$_i$ in an online manner. More specifically, at each t, for each i, we observe input $\hat{a}_{i,t}$ (determined at Step-I) and $\tilde{g}_{i,t}(\cdot)$, and need to determine $v_{i,t}$ irrevocably. In terms of feasibility, an immediate advantage is that it satisfies the inventory constraint (2) if it is a feasible solution to OOIC-S$_i$. However, we need further care to ensure the satisfaction of the allowance constraint (3) and allocation rate limit (4). As for performance guarantees, we can first analyze the performance of each step and then combine them to show the overall competitive analysis. In the following, we will discuss how the proposed online algorithm behaviors at both steps to ensure the feasibility and achieve a close-to-optimal CR.

5.2 The A&P$_l(\pi)$ Algorithm for General N

In this subsection, we will propose an online algorithm for a general number of inventories following the divide-and-conquer structure discussed in Sec. 5.1. We denote the algorithm as A&P$_l(\pi)$, where π is a parameter to be specified. In the following, we first introduce Step-II of A&P$_l(\pi)$, determining the allocation of OOIC-S$_i$ given the allowance from Step-I. We then introduce Step-I, determining the allowance of each inventory. We denote the allocated allowance from Step-I as $\hat{a}_{i,t}, \forall i, t$.

5.2.1 Step-II of A&P$_l(\pi)$. In this step, A&P$_l(\pi)$ determines the online inventory allocation for each OOIC-S$_i$ given the allocated allowance (denoted as $\{\hat{a}_{i,t}\}_{i\in[N]}$) from Step-I.

In Step-II of A&P$_l(\pi)$, it sets

$$\tilde{g}_{i,t}(v_{i,t}) = \pi \cdot g_{i,t}(v_{i,t}/\pi). \tag{24}$$

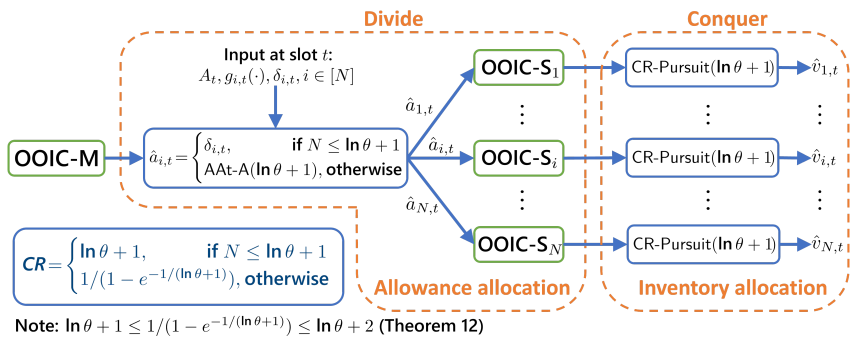


Fig. 1. An illustration of our divide-and-conquer approach and results.

We denote the optimal objective of OOIC-S$_i$ given its input up to slot t as $\tilde{OPT}_{i,t}$. We set $\tilde{OPT}_{i,0} = 0$. At each slot t, it determines the allocation $\hat{v}_{i,t}$ such that it satisfies

$$g_{i,t}(\hat{v}_{i,t}) = \frac{1}{\pi}\left(\tilde{OPT}_{i,t} - \tilde{OPT}_{i,t-1}\right). \tag{25}$$

While it looks similar to the CR-Pursuit(π) algorithm discussed in Sec. 4 and [21], we note that a major difference is that we are using the original revenue function $g_{i,t}(\cdot)$ to pursue a fraction of $1/\pi$ of the optimal objective achieved over the revenue function $\tilde{g}_{i,t}(\cdot)$ instead of $g_{i,t}(\cdot)$. In general, $\tilde{g}_{i,t}$, defined in (24), is no less than $g_{i,t}(\cdot)$ (take equality under the linear function case). Thus, it may be more difficult for Step-II to achieve the same performance ratio $(\ln\theta + 1)$ between the optimal objective to the online objective for each OOIC-S$_i$ as that in Sec. 4. However, we would show the performance ratio remains achievable in Lemma 7. The design of $\tilde{g}_{i,t}(\cdot)$ is important for achieving a better approximation ratio between the total optimal objective of the subproblems and the optimal objective of OOIC-M, which we will discuss in Step-I, Sec. 5.2.2.

To analyze Step-II, we first propose the following proposition on the properties of the online allocation $\hat{v}_{i,t}, \forall i, t$.

Lemma 6. *We have* $\hat{v}_{i,t} \leq \frac{1}{\pi} \cdot \hat{a}_{i,t}, \forall i, t$.

Proof. We have

$$g_{i,t}(\hat{v}_{i,t}) = \frac{1}{\pi}\left(\tilde{OPT}_{i,t} - \tilde{OPT}_{i,t-1}\right) \leq \frac{1}{\pi} \cdot \pi \cdot g_{i,t}(\hat{a}_{i,t}/\pi) \leq g_{i,t}(\hat{a}_{i,t}/\pi). \tag{26}$$

Thus, as $g_{i,t}(\cdot)$ is increasing, we conclude $\hat{v}_{i,t} \leq \frac{1}{\pi} \cdot \hat{a}_{i,t}$.

While this is a simple observation on the online solution, it plays an important role in designing the allowance allocation in Step-I, Sec. 5.2.2 and improving the overall performance of our online algorithm A&P$_l(\pi)$ for OOIC-M.

We denote the objective value of the online solution to OOIC-S$_i$ at slot t as $\eta_{i,t}$,

$$\eta_{i,t} \triangleq \sum_{\tau=1}^{t} g_{i,\tau}(\hat{v}_{i,\tau}) \tag{27}$$

We provide the performance analysis of Step-II in the following lemma. In particular, we choose π as $\ln\theta + 1$.

Lemma 7. *We have that for each $i \in [N]$, Step-II of $A\&P_l(\ln\theta+1)$ always produces a feasible solution to OOIC-S_i, and for any slot t, the online objective*

$$\eta_{i,t} \geq \frac{1}{\ln\theta+1} \cdot \tilde{OPT}_{i,t}. \tag{28}$$

Proof. The performance guarantee in (28) is implied by (25) when choosing $\pi = \ln\theta + 1$.

We now show the feasibility. We note that, when choosing $\pi = \ln\theta + 1$, OOIC-S_i with $\tilde{g}_{i,t}(\cdot)$ determined by (24), and a factor of $\frac{1}{\ln\theta+1}$ in objective value is equivalent to

$$\text{R-OOIC-S}_i: \ \max \sum_t g_{i,t}(z_{i,t}) \tag{29}$$

$$s.t. \sum_t z_{i,t} \leq \frac{C_i}{\ln\theta+1} \tag{30}$$

$$0 \leq z_{i,t} \leq \frac{\hat{a}_{i,t}}{\ln\theta+1}, \forall t, \tag{31}$$

where $z_{i,t}$, $v_{i,t}/(\ln\theta+1)$. Then, determining the online allocation according to (25) is equivalent to find $\hat{v}_{i,t}$ such that

$$g_{i,t}(\hat{v}_{i,t}) = \hat{OPT}_{i,t} - \hat{OPT}_{i,t-1}, \tag{32}$$

where $\hat{OPT}_{i,t}$ is the optimal objective of R-OOIC-S_i at slot t. It is clear that $\hat{v}_{i,t} \leq a_{i,t}/(\ln\theta+1)$ as $\hat{OPT}_{i,t} - \hat{OPT}_{i,t-1} \leq g_{i,t}(\hat{a}_{i,t}/(\ln\theta+1))$. Thus, the rate limit constraint in OOIC-S_i is satisfied.

We note that the R-OOIC-S_i is a single inventory problem we discuss in Sec. 4 with inventory capacity of $C_i/(\ln\theta+1)$. The online decision we make according to (32) suggests that we are running CR-Pursuit(1) over online R-OOIC-S_i. According to Lemma 4, for R-OOIC-S_i, we have

$$\sum_t \hat{v}_{i,t} \leq \frac{\ln\theta+1}{1} \cdot \frac{C_i}{\ln\theta+1} = C_i, \tag{33}$$

noting that the inventory capacity of R-OOIC-S_i equals $C_i/(\ln\theta+1)$. Thus, the online solution satisfies the capacity constraint in OOIC-S_i.

5.2.2 Step-I of $A\&P_l(\pi)$. Step-I is to determine $\{\hat{a}_{i,t}\}_{i\in[N]}$, the allowance allocation to different inventories at each slot t. Our goal is to determine an allocation such that we can guarantee a larger approximation ratio (, α) between $\sum_{i\in[N]} \tilde{OPT}_{i,t}$ and OPT_t at any slot t, i.e., $\sum_{i\in[N]} OPT_{i,t} \geq \alpha \cdot OPT_t$. Recall that $\tilde{OPT}_{i,t}$ is the optimal objective of OOIC-S_i up to slot t. And, OPT_t is the optimal objective of OOIC-M up to slot t.

As discussed in Sec. 5.1, we need further consideration to guarantee the satisfaction of the allowance constraint (3) and allocation rate limit (4). We characterize a sufficient condition on the allowance allocation such that the online solution $\{\hat{v}_{i,t}\}_{i\in[N]}$ determined at Step-II (as discussed in Sec. 5.2.1) satisfies constraints (3) and (4).

Lemma 8. *If the allowance allocation at each slot t satisfies*

$$\sum_{i\in[N]} \hat{a}_{i,t} \leq \pi \cdot A_t, 0 \leq \hat{a}_{i,t} \leq \pi \cdot \delta_{i,t}, \tag{34}$$

then the online solution $\{\hat{v}_{i,t}\}_{i\in[N]}$ determined by (25) *at Step-II satisfies the allowance constraint* (3) *and rate limit constraints* (4).

The idea is that according to Lemma 6, $\hat{v}_{i,t} \leq \hat{a}_{i,t}/\pi$. Together with (34), it implies that we have $\sum_{i\in[N]} \hat{v}_{i,t} \leq \sum_i \hat{a}_{i,t}/\pi \leq A_t$ and $\hat{v}_{i,t} \leq \delta_{i,t}$. Lemma 8 means that at each slot t, we can actually allocate π-time total allowance to the subproblems while guaranteeing the online solution satisfies constraints (3) and (4). We would show that it can help us significantly improve the approximation ratio α compared with allocating the allowance respecting constraints (3) and (4) directly.

In the online literature, the Step-I problem shares a similar setting as the free disposal model discussed in the online ads allocation problem in [19]. We defer the detailed discussion to Appendix C. We consider a generalized setting that, with Lemma 8, the online decision maker can allocate π-time more allowance with a π-time relaxer rate limit constraint at each slot than the offline optimal. We call it the *allowance augmentation scenario*. We note that this is different from other works in online literature, e.g., [23, 25, 32], where the authors consider the capacity augmentation scenario, i.e., how one can improve the performance guarantee (in particular, CR) of online algorithms when the online decision maker is equipped with more inventory capacity compared with the offline optimal. Here, to best of our knowledge, we are the first one to consider the allowance augmentation scenario. We provide our results in Theorem 9 and discuss how they generalize the existing studies afterward.

At Step-I of A&P$_l(\pi)$, we determine the allowance allocation by solving the following problem at each t. We call the problem AAt-A(π), standing for Allowance Allocation at slot t with Augmentation.

$$\text{AAt-A}(\pi): \quad \max \quad \sum_i \left(\tilde{g}_{i,t}(\hat{a}_{i,t}) - \int_0^{\hat{a}_{i,t}} \Psi_{i,t}(a)\, da\right) \tag{35}$$

$$\text{s.t.} \quad \sum_i \hat{a}_{i,t} \leq \pi \cdot A_t \tag{36}$$

$$0 \leq \hat{a}_{i,t} \leq \pi \cdot \delta_{i,t}, \forall i \in [N] \tag{37}$$

In AAt-A(π), $\{\hat{a}_{i,t}\}_{i\in[N]}$ is the allowance allocation at slot t. $\tilde{g}_{i,t}(\cdot)$ is defined as in (24). $\Psi_{i,t}(a)$ is defined as follows.

$$\Psi_{i,t}(a) = f_i(C_i) \cdot G_{i,t}(C_i, a) - \frac{1}{\pi \cdot C_i} \int_0^{C_i} G_{i,t}\,(x, a) \cdot f_i(x)\, dx. \tag{38}$$

where $f_i(x)$ and $G_i(x, a)$ are defined as,

$$f_i(x) = \frac{1}{\pi \cdot C_i} \frac{1}{e^{\frac{1}{\pi}} - 1} \cdot e^{x/(\pi \cdot C_i)}, \tag{39}$$

$$G_{i,t}\,(x, a) = \max \quad \sum_{\tau\in[t]} \tilde{g}_{i,\tau}(v_{i,\tau}) \tag{40}$$

$$\text{s.t.} \quad \sum_{\tau\in[t]} v_{i,\tau} \leq x \tag{41}$$

$$0 \leq v_{i,t} \leq a \tag{42}$$

$$0 \leq v_{i,\tau} \leq \hat{a}_{i,\tau}, \forall \tau \in [t-1]. \tag{43}$$

We can show that AAt-A(π) is a convex optimization problem with simple linear packing constraints by checking that $\Psi_{i,t}(a)$ is non-decreasing in a (shown in Proposition 16 in Appendix D). We can solve it using projected gradient descent where at each step, an evaluation of $\Psi_{i,t}(a)$ is required. Although we do not have a close form of $G_{i,t}(\cdot)$, we can evaluate $\Psi_{i,t}(a)$ efficiently using numerical integration methods.

Here, we discuss some understandings of the design of algorithm AAt-A(π). First, $G_{i,t}(x, a)$ defined in (40), is the optimal revenue of subproblem i with capacity x if it is allocated a allowance at slot t and given the past allowance allocation to subproblem i. It provides detailed information about the optimal inventory allocation of subproblem i, e.g., $\partial G_{i,t}(\cdot)/\partial x$ represents the marginal revenue at allocating the x capacity of inventory i. Second, the threshold $\Psi_{i,t}(a)$ when we allocate a allowance to inventory i depends on the detailed optimal revenue $G_{i,t}(x, a)$. This is because the allowance allocation would affect the optimal inventory allocation of the subproblem and thus the detailed optimal revenue. By comparison, the one in the primal-and-dual approach [22, 26], i.e., (8) in Sec. 3.3, only depends on the allocated amount as it directly allocates the inventory, and the newly allocated inventory would not impact the past allocation. Third, inspired by the exponential weighting algorithm in [19], we set the threshold as the exponential weighting average of the per-unit revenue of $G_{i,t}(x, a)$, where $f_i(x)$ is a carefully-designed weighting function. When $\pi = 1$ and restricting the revenue functions to be linear, AAt-A(π) can be viewed as a continuous (or fractional) counterpart of the exponential weighting approach in [19].

Also, to explore the allowance augmentation scenario under the concave revenue function case, we apply the following two novel ideas. First, we redesign the weighting function $f_{i,t}(\cdot)$, where we tune the weight according to the allowance augmentation level π. Second, due to the diminishing return effect of concave functions, the allowance augmentation may not provide substantial additional revenue to the online decision maker. To handle this problem, we design the revenue function $\tilde{g}_{i,t}(v_{i,t}) = \pi \cdot g_{i,t}(v_{i,t}/\pi)$ to ensure that increasing the allowance allocation for inventory i can substantially increase the revenue when compared with $g_{i,t}(v_{i,t})$ in the offline optimal. By comparison, in the linear function case, by allocating more allowance to inventory i, the revenue of the inventory could increase at a constant rate. In such a case, it is sufficient to adopt $g_{i,t}(\cdot)$ directly, and indeed we have $\tilde{g}_{i,t}(v_{i,t}) \equiv g_{i,t}(v_{i,t})$ following our design.

We can show the approximation guarantee of AAt-A(π) in the following theorem.

Theorem 9. *Given the allowance allocation $\hat{a}_{i,t}$ by solving AAt-A(π), we have*

$$\sum_{i \in [N]} \tilde{OPT}_{i,t} \geq \alpha(\pi) \cdot OPT_t, \tag{44}$$

where $\alpha(\pi) = \pi \cdot (1 - e^{-1/\pi})$. Furthermore, $\alpha(\pi)$ equals $\frac{e-1}{e}$ when $\pi = 1$ and 1 when $\pi \to \infty$.

The proof of Theorem 9 is provided in Appendix D. Our proof follows the online primal-and-dual analysis in [13, 19]. We use the dual problem of OOIC-M as a baseline for comparison. By carefully designing or updating the dual variable at each slot, we show that our increment in the total optimal objective of all subproblems OOIC-S$_i$ is at least a fraction of $\alpha(\pi)$ of the increment on the objective of dual OOIC-M at each slot t. This directly leads to Theorem 9.

Remarks. The results we show in Theorem 9 are with broader application scenarios and of independent interest. When all the revenue function is linear with a constant slope, i.e., all inventories have a uniform unit price, the Step-I problem reduces to maximizing the total amount of allocation, which is studied in [23, 24]. Our result (Theorem 9) implies that when there is no allowance augmentation (i.e., $\pi = 1$), it reproduces the competitive ratio $\frac{e}{e-1}$ as shown in [23, 24]. Also, when restricting to the linear function case and fixing $\pi = 1$, the Step-I problem can be viewed as a continuous counterpart of the online ad allocation problem with free disposal studied in [19]; see more details in Appendix C. In such case, we recover the competitive ratio $\frac{e}{e-1}$.

In both cases, our results, allowing $\pi \geq 1$, generalize to the online allowance augmentation case, where we can allocate π-time more amount of allowance (and subject to the π-time relaxer rate limit constraints) than the offline does. And, we show an improved CR of $1/(1 - e^{-1/\pi})/\pi$ with π-time augmentation, which tends to one when $\pi \to \infty$, as discussed in Theorem 9.

We also note that Theorem 9 holds for arbitrary increasing and differential concave functions starting from the origin, not restricted to the revenue functions we consider in set $\mathcal{G}$. This would be useful for generalizing our approaches to a broader application area with different sets of revenue functions beyond $\mathcal{G}$, which we will discuss in Sec. 6.

5.2.3 Competitive analysis of A&P$_l(\pi)$. We first summarize A&P$_l(\pi)$. At each slot t, (Step-I) it solves AAt-A(π) to obtain the allowance allocation $\hat{a}_{i,t}, \forall i \in [N]$, and (Step-II) determines $\hat{v}_{i,t}$ according to (25), for all $i \in [N]$.

We then show its performance guarantee off OOIC-M in the following theorem. In particular, we choose π as $\ln\theta + 1$.

Theorem 10. *The A&P$_l$($\ln\theta + 1$) algorithm is $1/(1 - e^{-1/(\ln\theta+1)})$-competitive for OOIC-M.*

Proof. We first show the feasibility of A&P$_l$($\ln\theta + 1$). By solving AAt-A($\ln\theta + 1$), $\{\hat{a}_{i,t}\}_{i\in[N],t\in[T]}$ satisfies condition (34) with $\pi = \ln\theta + 1$ in Lemma 8. According to Lemma 8, the online solution satisfies the allowance constraint and rate limit constraint of OOIC-M. Besides, according to Lemma 7, the online solution is always feasible to OOIC-S$_i$, i.e., it satisfies the capacity constraint of OOIC-M. We conclude the online solution ofA&P$_l$($\ln\theta + 1$) is feasible.

As for the CR, combining Theorem 9 we obtain in Step-I of A&P$_l$($\ln\theta + 1$) and Lemma 7 in Step-II, we have that the online objective of A&P$_l$($\ln\theta + 1$),

$$\sum_{i\in[N]} \eta_{i,t} \geq \frac{1}{\ln\theta + 1} \cdot \sum_{i\in[N]} \tilde{OPT}_{i,t} \geq \frac{1}{\ln\theta + 1}\alpha(\ln\theta + 1) \cdot OPT_t = (1 - e^{-1/(\ln\theta+1)}) \cdot OPT_t, \forall t. \quad (45)$$

Thus, at the final slot T, we also have $\sum_{i\in[N]} \eta_{i,T} \geq (1 - e^{-1/(\ln\theta+1)}) \cdot OPT_T$, and we conclude that A&P$_l$($\ln\theta + 1$) is $(1/(1 - e^{-1/(\ln\theta+1)}))$-competitive.

We note that $\ln\theta + 1 \leq 1/(1 - e^{-1/(\ln\theta+1)}) \leq \ln\theta + 2$. Thus, compared with the result under the single-inventory case shown in Theorem 5, Theorem 10 implies that we can achieve a CR with at most an additive constant (one) for the case with an arbitrary number of inventories. Also, the CR we achieve for OOIC-M under an arbitrary number of inventories is asymptotically equivalent to the one under the single-inventory case when $\theta \to \infty$.

5.3 A Simple Algorithm for Small N

From the design of our divide-and-conquer approach, we note that our online algorithm can allocate π-time more allowance to the subproblems according to Lemma 8. It reveals that when the number of the inventory is small (e.g., less than π), the allowance constraint could become redundant in our design. Leveraging the above insight, we show a simple and optimal online algorithm for OOIC-M when N is relatively small compared with θ. More specifically, we consider the case that $N \leq \ln\theta + 1$. We denote our online algorithm as A&P$_s(\pi)$ with π as a parameter to be specified. A&P$_s(\pi)$ consists of two steps, where the first step is to allocate the allowance, and the second step is to pursue a π performance ratio for each subproblem. In the first step, A&P$_s(\pi)$ determines the allowance allocation as

$$\hat{a}_{i,t} = \delta_{i,t}. \quad (46)$$

In the second step, for each OOIC-S$_i$, it chooses $\tilde{g}_{i,t}(v_{i,t})$ as $g_{i,t}(v_{i,t})$. We note that in such case OOIC-S$_i$ reduces to the single inventory problem we discuss in Sec. 4. The A&P$_s(\pi)$ determines the online solution running CR-Pursuit(π). That is, it chooses $\hat{v}_{i,t}$ such that

$$g_{i,t}(\hat{v}_{i,t}) = \frac{1}{\pi}(OPT_{i,t} - OPT_{i,t-1}), \quad (47)$$

where $OPT_{i,t}$ is the optimal objective of OOIC-S$_i$ given $\{\hat{a}_{i,\tau}\}_{\tau\in[t]}$ and $\{\tilde{g}_{i,t}(\cdot)\}_{\tau\in[t]}$ at slot t.

Theorem 11. *The A&P$_s$($\ln\theta + 1$) is $(\ln\theta + 1)$-competitive when $N \leq \ln\theta + 1$.*

Proof. We first check the feasibility of A&P($\ln\theta + 1$). The rate limit constraints and inventory constraints are directly guaranteed by the second step of A&P($\ln\theta + 1$), where we run the CR-Pursuit($\ln\theta + 1$) (as shown in Theorem 5). We then check the allowance constraints. For any t, we have

$$\sum_i \hat{v}_{i,t} \leq \sum_i \frac{1}{\ln\theta + 1} \cdot \hat{a}_{i,t} = \sum_i \frac{1}{\ln\theta + 1} \cdot \delta_{i,t} \leq \frac{1}{\ln\theta + 1} N \cdot A_t \leq A_t. \tag{48}$$

Recall that we have $\hat{v}_{i,t} \leq \hat{a}_{i,t}/(\ln\theta + 1)$ according to Lemma 2, and without loss of generality, we consider $\delta_{i,t} \leq A_t$, as discussed in Sec. 3.

We then show the performance analysis of the algorithm. It is clear that at each slot t, we have

$$\sum_i OPT_{i,t} \geq OPT_t, \forall t. \tag{49}$$

where OPT_t is the optimal objective of OOIC-M at slot t. This is because $\sum_i OPT_{i,t}$ equals the optimal objective of OOIC-M at slot t without the allowance constraint. Then, we have the online objective

$$\sum_{i,t} g_{i,t}(\hat{v}_{i,t}) = \frac{1}{\ln\theta + 1} \sum_i OPT_{i,T} \geq \frac{1}{\ln\theta + 1} OPT_T. \tag{50}$$

Thus, A&P($\ln\theta + 1$) is ($\ln\theta + 1$)-competitive.

Theorem 11 shows that when the total number of inventories is relatively small compared with the uncertainty range of the revenue functions (i.e., θ), we can reduce the multiple inventory problem to the single inventory case with the same performance guarantee.

5.4 Summary of Our Proposed Online Algorithm

Algorithm 1: A&P(π) Algorithm

```
1: At slot t, {g_{i,t}(·)}_{i∈[N]}, A_t, and {δ_{i,t}}_{i∈[N]} are revealed,
2: if N ≤ π then
3:     Run A&P_s(π), i.e., determine â_{i,t} = δ_{i,t} as in (46)
       and determine v̂_{i,t} according to (47), for all i ∈ [N],
4:     return {v̂_{i,t}}_{i∈[N]}.
5: else
6:     Run A&P_l(π):
7:     Step-I: solve AAt-A(π) to obtain the allowance allocation â_{i,t}, ∀i ∈ [N],
8:     Step-II: determine v̂_{i,t} according to (25), for all i ∈ [N],
9:     return {v̂_{i,t}}_{i∈[N]}.
10: end if
```

In this section, we summarize our online algorithm, denoted as A&P(π), and provide its performance analysis. An illustration of our approach and results is provided in Fig. 1. We also compare our results with existing ones in Fig 2.

The pseudocode of A&P(π) is provided in Algorithm 1. Depending on the value of N and θ, we run either A&P$_s(\pi)$ or A&P$_l(\pi)$. The CR of our online algorithm is shown in the following theorem.

Theorem 12. *Our online algorithm A&P($\ln\theta + 1$) achieves the following CR,*

$$\mathcal{CR}_1(A\&P(\ln\theta + 1)) = \begin{cases} \ln\theta + 1, & N \leq \ln\theta + 1 \\ 1/(1 - e^{-1/(\ln\theta+1)}), & otherwise. \end{cases} \tag{51}$$

Theorem 12 simply combines the results we show in Theorem 11 and Theorem 10. The CR we obtain is tight and optimal when N is smaller than $\ln\theta + 1$. This also recovers the results for the single inventory case discussed in Sec. 4 and [22]. It is within an additive constant of one to the lower bound when N is larger than $\ln\theta + 1$. When $\theta = 1$, our problem reduces to maximizing total allocation, and our result recovers the optimal CR $e/(e-1)$ achieved in [23, 24]. In [26], the authors show a CR that is within $[\ln\theta + 1, 1/(1 - e^{-1/(\ln\theta+1)})]$, independent of N, and consistently lower than the one achieved in [22]. They also show that the CR they achieve is tight when N tends to infinity. While our achieved CR at large N is worse than [26], the gap between them is no greater than an additive constant of one. In addition, we achieve a better (and optimal) CR when N is small. We provide an illustration of the CRs achieved by [22, 26] and ours in Fig. 2.

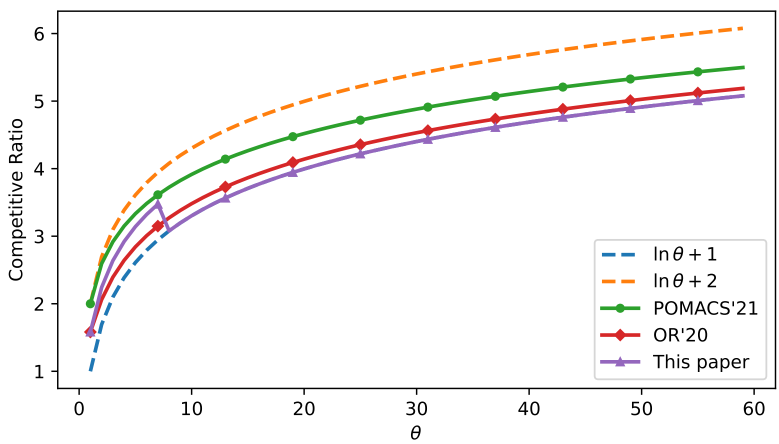


Fig. 2. The competitive ratios as a function of θ achieved by OR'20 [26], POMACS'21 [22], and this paper, under the setting of $N = 3$.

5.5 Numerical Comparison with State-of-the-art

	Online Revenue	Online-to-Offline Ratio	CR (or CR bound)
POMACS'21 [22]	7.020	3.191	3.670
OR'20 [26]	7.223	3.102	3.213
This paper	7.463	3.002	3.015

Table 3. The online-to-offline performance ratios of a particular input achieved by OR'20 [26], POMACS'21 [22], and this paper, under the setting of $N = 3$, $\theta = 7.5$, and $T = 200$. For this input, the performance ratios achieved by OR'20 [26] and POMACS'21 [22] are already higher than the optimal CR $\ln\theta + 1 \approx 3.015$. This implies that their achieved competitive ratios are sub-optimal in general.

As discussed in Sec. 5.4, the performance guarantee of our algorithm consists of two parts depending on the relative value of the number of inventories, N, and the parameter representing the level of revenue function uncertainty, θ. Here, we provide understandings of the performance of the state-of-the-art and our proposed algorithm in the above two cases. In our evaluation, we consider a scenario that the online decision maker has $N = 3$ inventories, each with a capacity $C_i = 1, \forall i$. We fix the minimum marginal revenue of the revenue function as $p_{\min} = 1$.

We first consider the case where $\ln\theta+1 \geq N$. Specifically, we choose $p_{\max} = 7.5$, and thus $\theta = 7.5$ with $\ln\theta + 1 \approx 3.015$. We fix the allowance allocation and rate limits to be one at each slot. We set the revenue functions among a period of $T = 200$ slots as linear functions with slopes increasing uniformly from $p_{\min}$ to $p_{\max}$. The results are illustrated in Table. 3. We observe that, for this input, the performance ratios achieved by OR'20 [26] and POMACS'21 [22] are already higher than the optimal CR $\ln\theta + 1 \approx 3.015$. This implies that their achieved competitive ratios are sub-optimal when the input uncertainty of the revenue functions is large.

We then discuss the case when θ is relatively small where we set $p_{\max} = 5$, and thus $\theta = 5$ with $\ln\theta + 1 \approx 2.609$. We set the revenue functions among a period of $T = 150$ slots as linear functions, the slopes of which are shown in Fig. 3a. The rate limits are randomly generated from $[0, 1]$, and the allowances are randomly generated from $[0, 3]$ for each slot (c.f. Fig. 3b). We compare our algorithm with OR'20 [26] and illustrate the revenue per slot and the accumulated revenue of the two algorithms in Fig. 3c. We observe that our algorithm is more conservative at the beginning, waiting for larger prices later. In contrast, OR'20 [26] uses the inventories more aggressively. The aggressive behavior of OR'20 leads to less remaining capacity at the end when high prices come. Meanwhile, the conservative decisions of our method at the beginning reward us with more capacity for higher prices and make our algorithm outperform state-of-the-art for this case. We understand that this is because the CR-Pursuit framework we apply in each subproblem maintains the online-to-offline ratio directly, while the primal-and-dual framework maintains the ratio between the online value to a dual one. As the dual value is an upper bound on the optimal offline optimal, the primal-and-dual framework tends to allocate the inventory more aggressively.

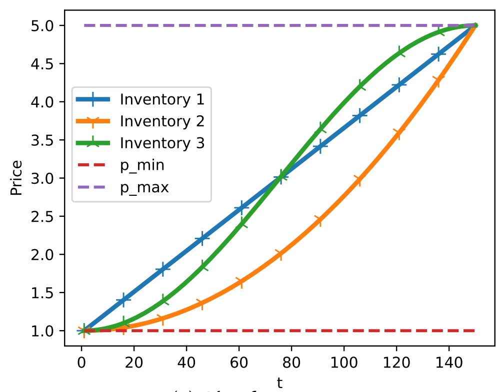
According to [21], due to the conservative nature of the CR-Pursuit algorithm, i.e., it stops allocating inventory as long as it achieves a fraction of $1/\pi$ revenue of the offline optimal, it may leave some inventory unutilized and fail to further increase the total revenue under non-worst case input. Here we also have a similar observation. In our second experiment, when the scopes of the linear functions are large at the beginning (instead of increasing slowly as in Fig. 3a), our algorithm tends to perform worse than the state-of-the-art ones. This motivates us to consider going beyond the worst-case analysis and providing improved average performance by utilizing the unallocated amount without sacrificing the worst-case guarantee, which we leave as future work.

6 EXTENSION TO GENERAL CONCAVE REVENUE FUNCTION

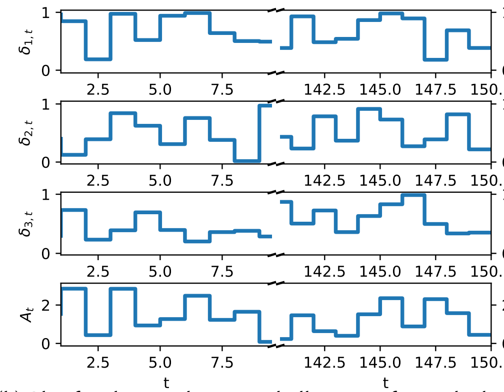
In addition to the set of revenue functions $\mathcal{G}$ we discussed above, our divide-and-conquer approach can be applied under a broader range of functions with corresponding applications. For example, we widely observe the logarithmic functions (e.g., $\log(v + 1)$) in wireless communication [33, 34], which is not covered by the revenue function set $\mathcal{G}$ when considering sufficiently large capacity. Also, the revenue functions in the application of one-way trading with price elasticity [21]. In general, let us consider a given set of concave revenue functions with zero value at the origin; say $\tilde{\mathcal{G}}$. We define $\Phi_{\tilde{\mathcal{G}}}(1)$ as the maximum online total allocation of running CR-Pursuit(1) under revenue functions $\tilde{\mathcal{G}}$ in the single inventory case (as defined in (17) with $\pi = 1$). It represents the maximum capacity we require to maintain the same performance of the offline optimal at all times. We have the following results for the OOIC-M under the set of revenue functions $\tilde{\mathcal{G}}$.

Proposition 13. *Suppose we can find $\tilde{\pi}$ such that for OOIC-S, we have*

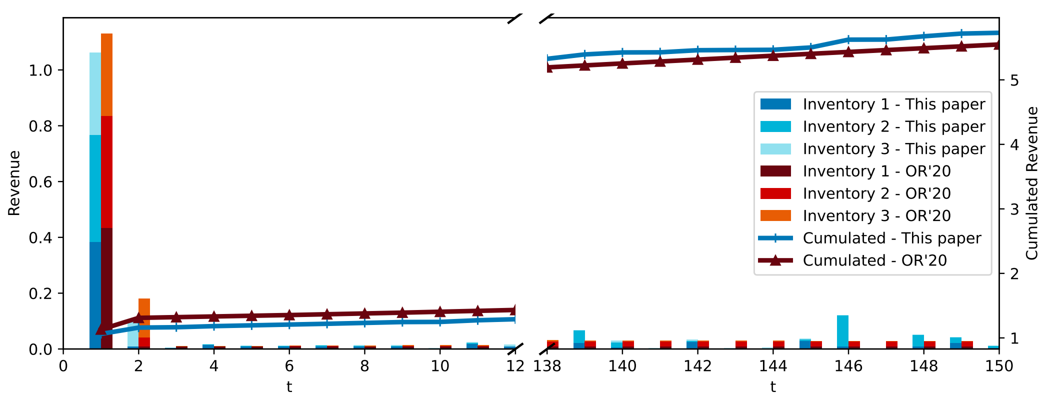
$$\Phi_{\tilde{\mathcal{G}}}(1) \leq \tilde{\pi} \cdot C. \tag{52}$$



(a) Plot for prices.



(b) Plot for the rate limits and allowance for each slot.



(c) Revenue per slot and cumulative revenue for the algorithms in OR'20 [26] and this paper.

Fig. 3. Numerical comparison between this paper and OR'20 [26] for $\theta = 5$, $N = 3$, $T = 150$ with different price sequences and rate limits.

We can run the A&P($\tilde{\pi}$) for OOIC-M under $\tilde{\mathcal{G}}$. The competitive ratio of A&P($\tilde{\pi}$) is given by,

$$\mathcal{CR}_{\tilde{G}}(A\&P(\tilde{\pi})) = \begin{cases} \tilde{\pi}, & N \leq \tilde{\pi}, \\ 1/(1-e^{-1/\tilde{\pi}}), & otherwise. \end{cases} \tag{53}$$

The proof follows the same idea as discussed in Sec. 5 and is omitted here. When $\tilde{\pi} < N$, we simply recover Lemma 7 with the condition (52). Together with Theorem 9, we show the results in (53) when $\tilde{\pi} \geq N$. As for the case $\tilde{\pi} < N$, we can recover Theorem 11 by noting that $\Phi_{\tilde{\mathcal{G}}}(\tilde{\pi}) \leq \frac{1}{\tilde{\pi}}\Phi_{\tilde{\mathcal{G}}}(1) \leq C$ (due to the concavity of the revenue functions), i.e., CR-Pursuit($\tilde{\pi}$) is feasible and $\tilde{\pi}$-competitive for OOIC-S.

For example, we can consider the one-way trading with price elasticity problem with multiple inventories, where the single-inventory case is proposed in [21]. More specifically, we consider the following type of revenue function, which we denote as $\hat{\mathcal{G}}$.

- $g_{i,t}(v_{i,t}) = \big(p_{i,t} - q_{i,t}(v_{i,t})\big) \cdot v_{i,t}$, $v_{i,t} \in [0, \delta_{i,t}]$, where $q_{i,t}(\cdot)$ is a convex increasing function with $q_{i,t}(0) = 0$ and $p_{i,t} \in [p_{\min}, p_{\max}]$. [1]

The revenue functions in $\hat{\mathcal{G}}$ consider that the price of selling (allocating) the inventory decreases as the supply (allocation) increases, which follows the basic law of supply and demand in microeconomics. In particular, the price elasticity is captured by a convex increasing function $q_{i,t}(\cdot)$, meaning that more supply would further decrease the price. The marginal revenue of the revenue function in $\hat{\mathcal{G}}$ is bounded between $p_{\min}$ and $p_{\max}$ only at the origin and could even be zero otherwise. It implies that problem OOIC-M under $\hat{\mathcal{G}}$ is not covered by [22]. Also, it can not be directly mapped to the discrete counterpart in [26], as we know when items in [26] could take values between 0 and $p_{\max}$, there is no finite competitive ratio for the problem. According to Lemma 15 in [21] (while it does not consider rate limit constraint, we can check that the proof simply follows with limit constraint), we have

$$\Phi_{\hat{G}}(1) \leq 2 \cdot (\ln\theta + 1) \cdot C. \tag{54}$$

For OOIC-M under revenue function set, $\hat{\mathcal{G}}$, we have

Proposition 14. *A&P($2 \cdot (\ln\theta + 1)$) achieves the following competitive ratio for OOIC-M under revenue function set $\hat{\mathcal{G}}$,*

$$\mathcal{CR}_{\hat{\mathcal{G}}}(A\&P(2 \cdot (\ln\theta + 1))) = \begin{cases} 2 \cdot (\ln\theta + 1), & N \leq 2 \cdot (\ln\theta + 1), \\ 1/(1 - e^{-1/(2\cdot(\ln\theta+1)}), & otherwise. \end{cases} \tag{55}$$

The CR we achieve is upper bounded by $2\ln\theta + 3$, which is up to a constant factor multiplying the lower bound $\ln\theta + 1$. This provides the first CR of the OOIC-M under revenue function set $\hat{\mathcal{G}}$ with application to the one-way trading with price elasticity under the multiple-inventory scenario. It is interesting to see whether we can fine a tiger bound on $\Phi_{\hat{\mathcal{G}}}(1)$ and achieve a better competitive ratio. In addition, while the determination of $\Phi_{\tilde{G}}(1)$ could be problem-specific, how we can show a general way for it would be another interesting future direction.

7 CONCLUDING REMARKS

In this work, we study the competitive online trading problem with multiple inventories, OOIC-M. The online decision maker allocates or trades the capacity-limited inventories to maximize the overall revenue, while the revenue functions and the allocation constraints at each slot come in an online manner. Our key result is a divide-and-conquer approach that reduces the multiple-inventory problem to solving multiple calibrated single-inventory problems. We optimize the allowance allocation among the subpoblems and combine their solutions. In particular, we show that the competitive ratio our approach achieves is optimal when N is small and is within an additive constant to the lower bound when N is larger, when considering gradient bounded revenue functions. We also provide a general condition for applying our approach to broader applications with different interesting sets of revenue functions. In particular, for revenue functions that appear in one-way trading with price elasticity, our approach obtains an optimal CR for the problem that is up to a constant factor to the lower bound. As a by-product, we also provide the first allowance augmentation results for the online fractional matching problem and the online fraction allocation problem with free disposal. As a future direction, we are interested in how our divide-and-conquer approach can be used to solve other online optimization problems with multi-entity and how to apply it in more application scenarios.

[1] There exist an v (may be infinity) such the function $g_{i,t}(\cdot)$ is increasing in $[0, v]$ and decreasing in $[v, \infty]$. We only need to consider the case that $\delta_{i,t} \leq v$ as no reasonable algorithm would allocate more than v at $g_{i,t}(\cdot)$. Thus, we consider that $g_{i,t}(\cdot)$ is increasing in $[0, \delta_{i,t}]$.

ACKNOWLEDGEMENT

The work presented in this paper was supported in part by a General Research Fund from Research Grants Council, Hong Kong (Project No. 11206821), an InnoHK initiative, The Government of the HKSAR, and Laboratory for AI-Powered Financial Technologies. Qiulin Lin would like to thank Dr. Hanling Yi and Dr. John Z. F. Pang for the discussions at the initial stage of the study.

REFERENCES

[1] Shizhen Zhao, Xiaojun Lin, and Minghua Chen. Robust online algorithms for peak-minimizing ev charging under multistage uncertainty. *IEEE Transactions on Automatic Control*, 62(11):5739–5754, 2017.

[2] Bahram Alinia, Mohammad Sadegh Talebi, Mohammad H Hajiesmaili, Ali Yekkehkhany, and Noel Crespi. Competitive online scheduling algorithms with applications in deadline-constrained ev charging. In *2018 IEEE/ACM 26th International Symposium on Quality of Service (IWQoS)*, pages 1–10. IEEE, 2018.

[3] Qiulin Lin, Hanling Yi, and Minghua Chen. Minimizing cost-plus-dissatisfaction in online ev charging under real-time pricing. *IEEE Transactions on Intelligent Transportation Systems*, 2021.

[4] Shuoyao Wang, Suzhi Bi, Ying-Jun Angela Zhang, and Jianwei Huang. Electrical vehicle charging station profit maximization: Admission, pricing, and online scheduling. *IEEE Transactions on Sustainable Energy*, 9(4):1722–1731, 2018.

[5] Lian Lu, Jinlong Tu, Chi-Kin Chau, Minghua Chen, and Xiaojun Lin. Online energy generation scheduling for microgrids with intermittent energy sources and co-generation. *ACM SIGMETRICS Performance Evaluation Review*, 41(1):53–66, 2013.

[6] Ying Zhang, Mohammad H Hajiesmaili, Sinan Cai, Minghua Chen, and Qi Zhu. Peak-aware online economic dispatching for microgrids. *IEEE transactions on smart grid*, 9(1):323–335, 2016.

[7] Yanfang Mo, Qiulin Lin, Minghua Chen, and S. Joe Qin. Optimal online algorithms for peak-demand reduction maximization with energy storage. In *Proc. ACM e-Energy*, pages 73–83, 2021.

[8] Yanfang Mo, Qiulin Lin, Minghua Chen, and S. Joe Qin. Optimal peak-minimizing online algorithms for large-load users with energy storage. In *IEEE INFOCOM*, 2021. poster paper.

[9] Minghong Lin, Adam Wierman, Lachlan LH Andrew, and Eno Thereska. Dynamic right-sizing for power-proportional data centers. *IEEE/ACM Transactions on Networking*, 21(5):1378–1391, 2012.

[10] Tan Lu, Minghua Chen, and Lachlan LH Andrew. Simple and effective dynamic provisioning for power-proportional data centers. *IEEE Transactions on Parallel and Distributed Systems*, 24(6):1161–1171, 2012.

[11] Ming Shi, Xiaojun Lin, Sonia Fahmy, and Dong-Hoon Shin. Competitive online convex optimization with switching costs and ramp constraints. In *IEEE INFOCOM 2018-IEEE Conference on Computer Communications*, pages 1835–1843. IEEE, 2018.

[12] Linqi Guo, John Pang, and Anwar Walid. Joint placement and routing of network function chains in data centers. In *IEEE INFOCOM 2018-IEEE Conference on Computer Communications*, pages 612–620. IEEE, 2018.

[13] Niv Buchbinder and Joseph Naor. *The design of competitive online algorithms via a primal-dual approach*. Now Publishers Inc, 2009.

[14] Aranyak Mehta. Online matching and ad allocation. *Foundations and Trends in Theoretical Computer Science*, 8(4):265–368, 2013.

[15] Yunhong Zhou, Deeparnab Chakrabarty, and Rajan Lukose. Budget constrained bidding in keyword auctions and online knapsack problems. In *International Workshop on Internet and Network Economics*, pages 566–576. Springer, 2008.

[16] Ran El-Yaniv, Amos Fiat, Richard M Karp, and Gordon Turpin. Optimal search and one-way trading online algorithms. *Algorithmica*, 30(1):101–139, 2001.

[17] Minghong Lin, Adam Wierman, Alan Roytman, Adam Meyerson, and Lachlan LH Andrew. Online optimization with switching cost. *ACM SIGMETRICS Performance Evaluation Review*, 40(3):98–100, 2012.

[18] Nikhil Bansal, Anupam Gupta, Ravishankar Krishnaswamy, Kirk Pruhs, Kevin Schewior, and Cliff Stein. A 2-competitive algorithm for online convex optimization with switching costs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2015)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2015.

[19] Jon Feldman, Nitish Korula, Vahab Mirrokni, S. Muthukrishnan, and Martin Pál. Online ad assignment with free disposal. In Stefano Leonardi, editor, *Internet and Network Economics*, pages 374–385, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.

[20] Nikhil R. Devanur and Kamal Jain. Online matching with concave returns. In *Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing*, STOC '12, page 137–144, New York, NY, USA, 2012. Association for

Computing Machinery.
[21] Qiulin Lin, Hanling Yi, John Pang, Minghua Chen, Adam Wierman, Michael Honig, and Yuanzhang Xiao. Competitive online optimization under inventory constraints. *Proc. ACM Meas. Anal. Comput. Syst.*, 3(1), March 2019.
[22] Bo Sun, Ali Zeynali, Tongxin Li, Mohammad Hajiesmaili, Adam Wierman, and Danny H.K. Tsang. Competitive algorithms for the online multiple knapsack problem with application to electric vehicle charging. *Proc. ACM Meas. Anal. Comput. Syst.*, 4(3), December 2020.
[23] Yossi Azar and Arik Litichevskey. Maximizing throughput in multi-queue switches. *Algorithmica*, 45(1):69–90, 2006.
[24] Bala Kalyanasundaram and Kirk R. Pruhs. An optimal deterministic algorithm for online b-matching. *Theoretical Computer Science*, 233(1):319–325, 2000.
[25] Han Deng and I-Hong Hou. Optimal capacity provisioning for online job allocation with hard allocation ratio requirement. *IEEE/ACM Transactions on Networking*, 26(2):724–736, 2018.
[26] Will Ma and David Simchi-Levi. Algorithms for online matching, assortment, and pricing with tight weight-dependent competitive ratios. *Operations Research*, 68(6):1787–1803, 2020.
[27] Nikhil R. Devanur, Kamal Jain, and Robert D. Kleinberg. Randomized primal-dual analysis of ranking for online bipartite matching. SODA '13, page 101–107, USA, 2013. Society for Industrial and Applied Mathematics.
[28] Rad Niazadeh and Robert D. Kleinberg. A unified approach to online allocation algorithms via randomized dual fitting. *CoRR*, abs/1308.5444, 2013.
[29] Julian Lorenz, Konstantinos Panagiotou, and Angelika Steger. Optimal algorithms for k-search with application in option pricing. *Algorithmica*, 55(2):311–328, 2009.
[30] Richard M Karp, Umesh V Vazirani, and Vijay V Vazirani. An optimal algorithm for on-line bipartite matching. In *Proceedings of the twenty-second annual ACM symposium on Theory of computing*, pages 352–358, 1990.
[31] Yossi Azar, Niv Buchbinder, TH Hubert Chan, Shahar Chen, Ilan Reuven Cohen, Anupam Gupta, Zhiyi Huang, Ning Kang, Viswanath Nagarajan, Joseph Naor, et al. Online algorithms for covering and packing problems with convex objectives. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 148–157. IEEE, 2016.
[32] Han Deng, Tao Zhao, and I-Hong Hou. Online routing and scheduling with capacity redundancy for timely delivery guarantees in multihop networks. *IEEE/ACM Transactions on Networking*, 27(3):1258–1271, 2019.
[33] Niv Buchbinder, Liane Lewin-Eytan, Ishai Menache, Joseph Naor, and Ariel Orda. Dynamic power allocation under arbitrary varying channels—an online approach. *IEEE/ACM Transactions on Networking*, 20(2):477–487, 2012.
[34] Niv Buchbinder, Liane Lewin-Eytan, Ishai Menache, Joseph Naor, and Ariel Orda. Dynamic power allocation under arbitrary varying channels - the multi-user case. In *2010 Proceedings IEEE INFOCOM*, pages 1–9, 2010.

APPENDIX

A PROOF OF PROPOSITION 1

Our proof follows the well-established online primal-and-dual approach [13, 22, 26, 28], etc.

We note that according to Appendix E of [26]. The threshold function is increasing and satisfies the following conditions.

$$C_i\phi_i'(w) - \phi_i(w) \le p_{\min}(\tilde{\chi} - 1),\ w \in (0, \chi \cdot C_i); \tag{56}$$

$$C_i\phi_i'(w) - \tilde{\chi} \cdot \phi_i(w) \le 0,\ w \in (\chi \cdot C_i, C_i). \tag{57}$$

Accordingly, it simply implies that

$$C_i\phi_i(w) - \int_0^w \phi_i(w)dw \le p_{\min}(\tilde{\chi} - 1) \cdot w,\ w \in (0, \chi \cdot C_i); \tag{58}$$

$$C_i\phi_i(w) - \int_0^w \phi_i(w) \le (\tilde{\chi} - 1) \cdot \left(p_{min} \cdot \chi \cdot C_i + \int_{\chi \cdot C_i}^w \phi_i(w)dw\right),\ w \in (\chi \cdot C_i, C_i). \tag{59}$$

In the primal-and-dual framework, it applies the dual problem as a baseline for the offline optimal. The dual problem of OOIC-M at slot T,

$$\text{Dual-OOIC-M}: \quad \min \sum_{i,t \in [T]} h_{i,t}(\alpha_i + \beta_t) + \sum_i C_i\alpha_i + \sum_t A_t\beta_t \tag{60}$$

$$\text{s.t.} \quad \alpha_i \ge 0, \beta_t \ge 0, \forall t, i, \tag{61}$$

where

$$h_{i,t}(\lambda) = \max_{0 \le v \le \delta_{i,t}} g_{i,t}(v) - \lambda \cdot v. \tag{62}$$

We denote the online solution of the algorithm as $\bar{v}_{i,t}$, which is the optimal solution to the problem (P&D) in (8). Recall that $w_{i,t}$ is the online total allocation of the algorithm from slot 1 to slot t, i.e.,

$$w_{i,t} = \sum_{\tau=1}^{t} \bar{v}_{i,\tau}. \tag{63}$$

At each slot t, we denote the optimal dual solution of the problem (P&D) in (8) associated with constraint (9) as $\hat{\beta}_t$. Note that according to KKT conditional, we have

$$\hat{\beta}_t \cdot \Big(\sum_{i \in [N]} \bar{v}_{i,t} - A_t\Big) = 0. \tag{64}$$

We consider the following dual solution,

$$\alpha_i = \phi_i(w_{i,T}), \forall i; \beta_t = \hat{\beta}_t, \forall t \in [T]. \tag{65}$$

We note that the dual variable satisfies the dual constraint (61). Then, we have

$$\begin{aligned}
OPT_T &\le \sum_{i \in [N], t \in [T]} h_{i,t}(\alpha_i + \beta_t) + \sum_{i \in [N]} C_i \alpha_i + \sum_{t \in [T]} A_t \beta_t && (66)\\
&= \sum_{i \in [N], t \in [T]} h_{i,t}(\phi_i(w_{i,T}) + \hat{\beta}_t) + \sum_{i \in [N]} C_i \phi_i(w_{i,T}) + \sum_{t \in [T]} A_t \hat{\beta}_t && (67)\\
&\overset{(a)}{\le} \sum_{i \in [N], t \in [T]} h_{i,t}(\phi_i(w_{i,t}) + \hat{\beta}_t) + \sum_{i \in [N]} C_i \phi_i(w_{i,T}) + \sum_{t \in [T]} A_t \hat{\beta}_t && (68)\\
&\overset{(b)}{\le} \sum_{i \in [N], t \in [T]} \Big[g_{i,t}(\bar{v}_{i,t}) - (\phi_i(w_{i,t}) + \hat{\beta}_t)\bar{v}_{i,t}\Big] + \sum_{i \in [N]} C_i \phi_i(w_{i,T}) + \sum_{t \in [T]} A_t \hat{\beta}_t && (69)\\
&\overset{(c)}{=} \sum_{i \in [N], t \in [T]} \Big[g_{i,t}(\bar{v}_{i,t}) - \phi_i(w_{i,t})\bar{v}_{i,t}\Big] + \sum_{i \in [N]} C_i \phi_i(w_{i,T}) && (70)\\
&\overset{(d)}{\le} \sum_{i \in [N], t \in [T]} g_{i,t}(\bar{v}_{i,t}) + \sum_{i \in [N]} \Big[C_i \phi_i(w_{i,T}) - \int_0^{w_{i,T}} \phi_i(w) dw\Big] && (71)\\
&\overset{(e)}{\le} \tilde{\chi} \cdot \sum_{i \in [N], t \in [T]} g_{i,t}(\bar{v}_{i,t}), && (72)
\end{aligned}$$

where (a) is due to the non-decreasing of $\phi_i(\cdot)$ and $h_{i,t}(\cdot)$ defined in (62); (b) is due to $\bar{v}_{i,t}$ is the optimal solution to (62) when $\lambda = \phi_i(w_{i,t}) + \hat{\beta}_t$ by checking that the KKT conditions of the problem (P&D) in (8); (c) is according to (64); (d) is according to $\phi_i(w_{i,t})\bar{v}_{i,t} \ge \int_{w_{i,t}-\bar{v}_{i,t}}^{w_{i,t}} \phi_i(w)dw$; (e) is by the fact that

$$C_i \phi_i(w_{i,T}) - \int_0^{w_{i,T}} \phi_i(w) dw \le (\tilde{\chi} - 1) \sum_{t \in [T]} g_{i,t}(\bar{v}_{i,t}). \tag{73}$$

We show (73) in the following. When $w_{i,T} \le \chi \cdot C_i$, it directly follows (58) and $\sum_{t \in [T]} g_{i,t}(\bar{v}_{i,t}) \ge p_{min} \cdot w_{i,T}$. When $w_{i,T} \ge \chi \cdot C_i$, it follows (59) and that

$$\sum_{t \in [T]} g_{i,t}(\bar{v}_{i,t}) \ge p_{\min} \cdot \chi \cdot C_i + \int_{\chi \cdot C_i}^{w_{i,T}} \phi_i(w) dw \tag{74}$$

by the fact that if $\bar{v}_{i,t} > 0$,

$$g'_{i,t}(v) \geq max\{p_{\min}, \phi_i(w_{i,t-1} + v)\}, \forall v \leq \bar{v}_{i,t}. \tag{75}$$

(75) is due to that $g'_{i,t}(\bar{v}_{i,t}) \geq max\{p_{\min}, \phi_i(w_{i,t})\}$, which follows the concavity of $g_{i,t}(\cdot)$, non-decreasing property of $\phi_i(\cdot)$, and $g'_{i,t}(\bar{v}_{i,t}) \geq \phi_i(w_{i,t})$, if $\bar{v}_{i,t} > 0$ by the KKT condition of (P&D) in (8).

B PROOF OF LEMMA 4

We first consider a new class of revenue function,

- $g_t(v_t)$ is concave, increasing and differentiable with $g_t(0) = 0$;
- $\frac{g'(0)}{g_t(\delta_t)/\delta_t} \leq \xi$;
- $g'_t(0) \in [p_{\min}, p_{\max}]$.

where ξ is a given parameter. We denote this class of revenue function as Class_ξ. We can generalize the results (Theorem 8) in [21] by taking the rate limit constraint into consideration and obtain the following proposition

Proposition 15. *For Class_ξ revenue function, we have*

$$\Phi_\xi(\pi) \leq \xi \cdot (\ln\theta + 1) \cdot \frac{C}{\pi}. \tag{76}$$

We omit the detailed proof here as it is by simply checking the proof in [21] step-by-step when considering revenue function $g_t(v_t)$ defined over $v_t \in [0, \delta_t]$ instead of $v_t \in [0, C]$.

We now turn to the proof of Lemma 4.

Proof of Lemma 4. We prove the lemma by showing that for any $\epsilon > 0$, $\Phi_1(\pi) \leq (1+\epsilon)(\ln\theta + 1) \cdot C/\pi$.

For any function $g_t(v_t), v_t \in [0, \delta_t]$, we can construct a sequence of functions as follows. We begin by finding the maximum $v^{(1)} \leq \delta_t$ such that $g'_t(v^{(1)}) \geq g'_t(0)/(1+\epsilon)$. We define $g_t^{(1)}(v) = g_t(v), v \in [0, v^{(1)}]$. We then find the maximum $v^{(2)} \leq \delta_t$ such that $g'_t(v^{(2)}) \geq g'_t(v^{(1)})/(1+\epsilon)$. We define $g_t^{(2)}(v) = g_t(v^{(1)} + v) - g_t(v^{(1)}), v \in [0, v^{(2)} - v^{(1)}]$. We continue the steps until we arrive at δ_t. Suppose in total there are k_t functions we construct, and they are $\{g^{(i)}(v)\}_{i\in k_t}$, where $g^{(i)}(v) = g_t(v^{(i-1)} + v) - g_t(v^{(i-1)}), v \in [0, v^{(i)} - v^{(i-1)}]$. Also, $v^{(k)} = \delta_t$. We can easily check that $\{g^{(i)}(v)\}_{i\in k_t}$ belongs to $\text{Class}_{\xi=1+\epsilon}$.

For any $\sigma \in \Sigma$, suppose there is a revenue function $g_t(v_t)$ not belonging to $\text{Class}_{\xi=1+\epsilon}$, we can construct $\{g^{(i)}(v)\}_{i\in k_t}$ following the above procedure and replace $g_t(v_t)$ in σ. We denote the new input as $\tilde{\sigma}$. We can show that the replacement will not decrease the total allocation of CR-Pursuit(π), We note that the output of CR-Pursuit(π) does not change at $\tau < t$ as the venue function and increment of the optimal objective remains the same at those slots. Thus it is sufficient to show that

$$\hat{v}_t \leq \sum_{i=1}^{k_t} \tilde{v}_t^{(i)}, \tag{77}$$

where $\hat{v}_t$ is the output of CR-Pursuit(π) under σ at slot t and $\tilde{v}_t^{(i)}$ is the output of CR-Pursuit(π) under $\tilde{\sigma}$ at function $g_t^{(i)}(\cdot)$, for $i \in [k_t]$. Following the CR-Pursuit(π) algorithm, we have

$$g_t(\hat{v}_t) = \frac{1}{\pi}(OPT(t) - OPT(t-1)) = \sum_{i=1}^{k_t} g_t^{(i)}(\tilde{v}_t^{(i)}). \tag{78}$$

According to our construction and concavity of $g_t(\cdot)$, we have

$$\sum_{i=1}^{k_t} g_t^{(i)}(\tilde{v}_t^{(i)}) = \sum_{i=1}^{k_t} \left[g_t(v^{(i-1)} + \tilde{v}_t^{(i)}) - g_t(v^{(i-1)}) \right] \tag{79}$$

$$\overset{(a)}{\leq} \sum_{i=1}^{k_t} \left[g_t(\sum_{j=1}^{i-1} \tilde{v}_t^{(j)} + \tilde{v}_t^{(i)}) - g_t(\sum_{j=1}^{i-1} \tilde{v}_t^{(j)}) \right] \tag{80}$$

$$= g_t(\sum_{i=1}^{k_t} \tilde{v}_t^{(i)}), \tag{81}$$

where (a) is due to the concavity of $g_t(\cdot)$ and the fact that $\sum_{j=1}^{i-1} \tilde{v}_t^{(j)} \leq \sum_{j=1}^{i-1} (v^{(j)} - v^{(j-1)}) \leq v^{(i-1)}, \forall i \in [t_k]$. Thus, we have $g_t(\hat{v}_t) \leq g_t(\sum_{i=1}^{k_t} \tilde{v}_t^{(i)})$ and conclude (77).

This directly implies that we can replace all functions by a sequence of $\text{Class}_{\xi=1+\epsilon}$ functions without decreasing the total allocation of CR-Pursuit(π). Thus, $\Phi(\pi) \leq \Phi_{\xi=1+\epsilon}(\pi) \leq (1+\epsilon) \cdot (\ln\theta + 1) \cdot C/\pi, \forall \epsilon > 0$. And, we conclude $\Phi(\pi) \leq (\ln\theta + 1) \cdot C/\pi$

C RELATION BETWEEN STEP-I AND THE FREE DISPOSAL MODEL

In the following, we first introduce the free disposal discussed in [19]. We then discuss how the Step-I problem relates to the free disposal model discussed in the online advertisement allocation problem [19].

In [19], the authors study the problem that the online decision maker allocates the ads impressions to a fixed group of advertisers. Each advertiser has a contract for a given number of impressions; say c_i for advertiser i. At each time, an impression appears, together with a set of weighted edges between the impression and the advertisers. Only when the advertiser is interested in the impression, there is an edge between them, and the weight of an edge represents the value of such impression to the advertiser. The goal is to maximize the total weight of the allocation. The authors consider a free disposal model that it allows the online decision maker to allocate more impressions than the number in the contract (i.e., c_i) to advertisers. But the value of the allocation for any advertiser i achieves only consists of the most valuable c_i impressions allocated to advertiser i. An online algorithm aims to find an allocation that achieves a close total value to the offline optimal.

When restricting to linear revenue functions and fixing $\pi = 1$, the Step-I problem can be viewed as a continuous (fractional) counterpart of the above online ad allocation problem with free disposal. First, two problems are both optimal allocation problems among a fixed group of inventories (cf. advertisers) subjecting to capacity constraints (cf. number of impressions in the contract), allowance constraints (cf. one impression per time), and rate limit constraints (cf. existence of an edge between an impression and an advertiser). Second, the online decision maker shares the same design goal under both problems. The goal is to find an online allowance allocation such that the sum of the optimal objectives of the subproblems (given the allocated allowance), defined in OOIC-S$_i$, i.e., $\tilde{OPT}_{i,t}, \forall i \in [N]$, is close to the offline optimal of OOIC-M. In the linear function case, $\tilde{g}_{i,t}(\cdot) \equiv g_{i,t}(\cdot)$. Thus, the optimal objective of the subproblem i equals the optimal revenue of the inventory i (given the allocated allowance), for all i, which corresponds to the value of an advertiser in [19]. That is, while the total allowance we allocate to inventory i may exceed its capacity C_i, but the total revenue of inventory i, $\tilde{OPT}_{i,t}$, is the total value of the most valuable C_i amount among all the allocated allowance to inventory i. Third, with $\pi = 1$, the online decision maker in Step-I can allocate the same amount of allowance (and subject to the same rate limit constraints) as the offline optimal does, which is the same as the setting of the online problem with free disposal studied in [19].

Here, we generalize the problems to the allowance augmentation scenario that compared with the offline optimal, the online decision maker can allocate π-time more allowance (and subject to a π-time relaxer rate limit constraint) at each slot, as we state in Lemma 8.

In terms of analysis results, we recover the competitive ratio $\frac{e}{e-1}$ when there is no allowance augmentation, i.e., $\pi = 1$. Further allowing $\pi \geq 1$, Our results generalize to the online allowance augmentation case, and we show an improved CR of $1/(1-e^{-1/\pi})/\pi$ with π-time augmentation, which tends to one when $\pi \to \infty$, as discussed in Theorem 9.

D PROOF OF THEOREM 9

We first show a useful proposition.

Proposition 16. *$\Psi_{i,t}(a)$ is non-decreasing in a.*

Proof. By integration by parts,

$$\int_0^{c_i} f_i(x)\frac{\partial G_{i,t}(x,a)}{\partial x}dx = \int_0^{c_i} f_i(x)dG_{i,t}(x,a) \tag{82}$$

$$= f_i(x)\cdot G_{i,t}(x,a)\Big|_0^{C_i} - \frac{1}{\pi\cdot C_i}\int_0^{C_i} f_i(x)G_{i,t}(x,a)\,dx \tag{83}$$

$$= f_i(C_i)G_{i,t}(C_i,a) - \frac{1}{\pi\cdot C_i}\int_0^{C_i} f_i(x)G_{i,t}(x,a)\,dx \tag{84}$$

$$= \Psi_{i,t}(a). \tag{85}$$

And according to the sensitivity analysis of the optimization problem defining $G_{i,t}(x,a)$, we have

$$\frac{\partial G_{i,t}(x,a)}{\partial x} = \eta^*. \tag{86}$$

where η^* is the optimal dual variable associated with constraint (41). We can check that η^* is non-decreasing in a by the KKT condition, and thus $\Psi_{i,t}(a)$ is non-decreasing in a.

Proof of Theorem 9. We adapt the primal-and-dual framework [13, 19] to prove the theorem. We begin with the revenue increment of our algorithm AAt(π)-A at slot t, denoted as ΔP. According to (40),

$$\Delta P \triangleq \sum_i \left(\tilde{OPT}_{i,t} - \tilde{OPT}_{i,t-1}\right) = \sum_i \left(G_{i,t}(C_i,\hat{a}_{i,t}) - G_{i,t}(C_i,0)\right). \tag{87}$$

We note that the optimal solution of AAt(π)-A satisfies the KKT condition,

$$\tilde{g}'_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t}) - \tilde{\beta}_t - \tilde{\gamma}_{i,t} + \tilde{\sigma}_{i,t} = 0, \forall i \tag{88}$$

$$\tilde{\beta}_t\left(\sum_i \hat{a}_{i,t} - \pi\cdot A_t\right) = 0, \tag{89}$$

$$\tilde{\sigma}_{i,t}\cdot\hat{a}_{i,t} = 0, \forall i, \tag{90}$$

$$\tilde{\gamma}_{i,t}\left(\hat{a}_{i,t} - \pi\cdot\delta_{i,t}\right) = 0, \forall i, \tag{91}$$

where $\tilde{\beta}_t \geq 0$ and $\tilde{\sigma}_t^i \geq 0, \tilde{\gamma}_t^i \geq 0, \forall i$ are dual variables corresponding to constraints (36) and (37).

We first write down the dual problem of OOIC-M at slot t,

$$\text{Dual-OOIC-M}: \quad \min \sum_{i,\tau\in[t]} h_{i,\tau}(\alpha_i + \beta_\tau) + \sum_i C_i\alpha_i + \sum_\tau A_\tau\beta_\tau \tag{92}$$

$$\text{s.t.} \quad \alpha_i \geq 0, \beta_t \geq 0, \forall t, i, \tag{93}$$

where

$$h_{i,\tau}(\lambda) = \max_{0\leq v\leq\delta_{i,\tau}} g_{i,\tau}(v) - \lambda\cdot v. \tag{94}$$

We compare our online increment with the following dual solution. At slot t, we update the dual variable $\{\alpha_{i,t}\}_{i\in[N]}$, determine the dual variable β_t,

$$\alpha_{i,t} = \Psi_{i,t}(\hat{a}_{i,t}), \forall i; \beta_t = \tilde{\beta}_t. \tag{95}$$

We note that the dual variable satisfies the dual constraint (93).

Let the increment of the dual objective by the above dual solutions at each slot as ΔD. To prove the theorem, the most important step in the framework is to show that at each slot, we have

$$\Delta D \leq \frac{1}{\pi}\frac{1}{1-e^{-1/\pi}}\Delta P. \tag{96}$$

We can now compute the increment of the dual,

$$\begin{aligned}\Delta D = &\sum_{i,\tau<t}\left(h_{i,\tau}\left(\alpha_{i,t}+\beta_\tau\right) - h_{i,\tau}\left(\alpha_{i,t-1}+\beta_\tau\right)\right)\\ &+\sum_i h_{i,t}\left(\alpha_{i,t}+\beta_t\right) + \sum_i C_i\left(\alpha_{i,t}-\alpha_{i,t-1}\right) + \beta_t A_t \qquad (97)\\ \overset{(a)}{\leq}& \sum_i h_{i,t}\left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right) + \sum_i C_i\left(\Psi_{i,t}(\hat{v}_{i,t}) - \Psi_{i,t}(0)\right) + \tilde{\beta}_t A_t \qquad (98)\\ \overset{(b)}{=}& \sum_i \frac{1}{\pi}\left[\tilde{g}_{i,t}(\hat{a}_{i,t}) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\hat{a}_{i,t}\right] + \sum_i C_i\left(\Psi_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(0)\right) + \tilde{\beta}_t A_t \qquad (99)\\ \overset{(c)}{=}& \sum_i C_i\left(\Psi_{i,t}\left(\hat{a}_{i,t}\right) - \Psi_{i,t}(0)\right) + \frac{1}{\pi}\sum_i\left(\tilde{g}_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t})\cdot\hat{a}_{i,t}\right). \qquad (100)\end{aligned}$$

We show the above equality of inequality one by one.

- (a) is according to (95) (the way to set the dual variables) and the facts that $a_{i,t} \geq a_{i,t-1}$ and $h_{i,t}(\lambda)$ is non-increasing in λ.
- (b) is according to the fact that by (88), (90) and (91), $\hat{a}_{i,t}$ is the optimal solution to

$$\max_{0\leq v\leq\pi\cdot\delta_{i,\tau}} \tilde{g}_{i,\tau}(v) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\cdot v. \tag{101}$$

Also, we have

$$\max_{0\le v\le \pi\cdot\delta_{i,\tau}} \tilde{g}_{i,\tau}(v) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\cdot v \tag{102}$$

$$= \max_{0\le v\le \pi\cdot\delta_{i,\tau}} \pi\cdot g_{i,\tau}(\frac{v}{\pi}) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\cdot v \tag{103}$$

$$=\pi\cdot \max_{0\le v\le \pi\cdot\delta_{i,\tau}} \pi\cdot g_{i,\tau}(\frac{v}{\pi}) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\cdot \frac{v}{\pi} \tag{104}$$

$$=\pi\cdot \max_{0\le v\le \delta_{i,\tau}} \pi\cdot g_{i,\tau}(v) - \left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right)\cdot v \tag{105}$$

$$=\pi\cdot h_{i,t}\left(\Psi_{i,t}(\hat{a}_{i,t}) + \tilde{\beta}_t\right). \tag{106}$$

- (c) is according to (89).

Recall (38) that

$$\Psi_{i,t}(a) = f_i(C_i)\cdot G_{i,t}(C_i,a) - \frac{1}{\pi\cdot C_i}\int_0^{C_i} G_{i,t}\,(x,a)\cdot f_i(x)dx.$$

Plugging $\Psi_{i,t}(\cdot)$, i.e., (38)), in ΔD, we have that

$$\Delta D = \sum_i \left\{C_i\cdot f_i(C_i)G_{i,t}\left(C_i,\hat{a}_{i,t}\right) - \frac{1}{\pi}\cdot\int_0^{C_i} f_i(x)G_{i,t}\left(x,\hat{a}_{i,t}\right)dx\right. \tag{107}$$

$$\left. -\left(f_i(C_i)G_{i,t}\,(C_i,0) - \frac{1}{\pi}\cdot\int_0^{C_i} f_i(x)G_{i,t}\,(x,0)\,dx\right) + \frac{1}{\pi}\cdot\left(\tilde{g}_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t})\cdot\hat{a}_{i,t}\right)\right\}. \tag{108}$$

By the fact that $C_i\cdot f_i(C_i) = \frac{1}{\pi}\frac{1}{1-e^{-1/\pi}}$ according to (39), we have

$$\begin{aligned}\Delta D =& \frac{1}{\pi}\frac{1}{1-e^{-1/\pi}}\sum_i\left[G_{i,t}\left(C_i,\hat{a}_{i,t}\right) - G_{i,t}\,(C_i,0)\right]\\ &+\sum_i\left[-\frac{1}{\pi}\cdot\int_0^{C_i} f_i(x)G_{i,t}\left(x,\hat{a}_{i,t}\right)dx + \frac{1}{\pi}\cdot\int_0^{C_i} f_i(x)G_{i,t}\,(x,0)\,dx\right]\\ &+\sum_i\frac{1}{\pi}\cdot\left(\tilde{g}_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t})\cdot\hat{a}_{i,t}\right).\end{aligned} \tag{109}$$

Comparing with (87), to show (96), is sufficient to show that

$$\tilde{g}_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t})\cdot\hat{a}_{i,t} \le \int_0^{C_i} f_i(x)G_{i,t}\left(x,\hat{a}_{i,t}\right)dx - \int_0^{C_i} f_i(x)G_{i,t}\,(x,0)\,dx. \tag{110}$$

We further have

$$\tilde{g}_{i,t}(\hat{a}_{i,t}) - \Psi_{i,t}(\hat{a}_{i,t})\cdot\hat{a}_{i,t} \tag{111}$$

$$\le \tilde{g}_{i,t}(\hat{a}_{i,t}) - \int_0^{\hat{a}_{i,t}}\Psi_{i,t}(a)da \tag{112}$$

$$\le \tilde{g}_{i,t}(\hat{a}_{i,t}) - \int_0^{\hat{a}_{i,t}}\left[f_i(C_i)\cdot G_{i,t}(C_i,a) - \frac{1}{C_i}\int_0^{C_i} G_{i,t}\,(x,a)\cdot f_i(x)dx\right]da \tag{113}$$

$$\overset{(a)}{=} \tilde{g}_{i,t}(\hat{a}_{i,t}) - \int_0^{\hat{a}_{i,t}}\int_0^{c_i}\frac{\partial G_{i,t}(x,a)}{\partial x}\cdot f_i(x)dxda \tag{114}$$

$$= \int_0^{c_i}\int_0^{\hat{a}_{i,t}}\tilde{g}'_{i,t}(a) - \frac{\partial G_{i,t}(x,a)}{\partial x}da\cdot f_i(x)dx, \tag{115}$$

where (a) is due to (85).

So, it reduces to show that

$$\int_0^{\hat{a}_{i,t}} \tilde{g}'_{i,t}(a) - \frac{\partial G_{i,t}(x,a)}{\partial x} da \le G_{i,t}\left(x, \hat{a}_{i,t}\right) - G_{i,t}\left(x, 0\right), \tag{116}$$

which is equivalent to

$$\int_0^{\hat{a}_{i,t}} \tilde{g}'_{i,t}(a) - \frac{\partial G_{i,t}(x,a)}{\partial x} da \le \int_0^{\hat{a}_{i,t}} \frac{\partial G_{i,t}(x,a)}{\partial a} da. \tag{117}$$

To show the above inequality, it is sufficient to show that,

$$\tilde{g}'_{i,t}(a) \le \frac{\partial G_{i,t}(x,a)}{\partial x} + \frac{\partial G_{i,t}(x,a)}{\partial a}. \tag{118}$$

To proceed, we recall that $G_{i,t}(x,a)$ is the optimal objective to the following problem,

$$G_{i,t}\left(x,a\right) = \max \quad \sum_{\tau \in [t]} \tilde{g}_{i,\tau}(v_{i,\tau}) \tag{119}$$

$$\text{s.t} \quad \sum_{\tau \in [t]} v_{i,\tau} \le x \tag{120}$$

$$0 \le v_{i,t} \le a \tag{121}$$

$$0 \le v_{i,\tau} \le \hat{a}_{i,\tau}, \forall \tau \in [t-1]. \tag{122}$$

Let η, ψ_t and ϕ_t, and $\{\psi_\tau\}_{\tau\in[t-1]}$ and $\{\phi_\tau\}_{\tau\in[t-1]}$ be the dual variable associated with (120), (121), and (122), respectively.

According to sensitivity analysis of convex program, we have

$$\frac{\partial G_{i,t}(x,a)}{\partial x} = \eta^*, \frac{\partial G_{i,t}(x,a)}{\partial a} = \phi_t^*. \tag{123}$$

According to the KKT condition of the optimal solution, we have

$$\tilde{g}'_{i,t}(a) \le \tilde{g}'_{i,t}(v^*_{i,t}) = \eta^* + \phi_t^* - \psi_t^* \le \eta^* + \phi_t^*, \tag{124}$$

where $v^*_{i,t}$, ϕ_t^*, ψ_t^*, η^* and ϕ_t^* represent the optimal primal solution and dual solution, and the first inequality is due to the fact that $\tilde{g}_{i,t}(\cdot)$ is concave and $v^*_{i,t} \le a$.

Combining (123) and (124), we conclude (118), which leads to (116) and (117). Combining (115) and (116), we conclude (110). Combining (110), (87) and (109), we easily conclude (96). Summing (96) over all time slots, we have

$$\sum_{i\in[N]} \tilde{OPT}_{i,t} \ge \pi \cdot (1 - e^{-1/\pi}) \cdot \text{Dual-OOIC-M} \ge \pi \cdot (1 - e^{-1/\pi}) \cdot OPT_t. \tag{125}$$

Received February 2022; revised March 2022; accepted April 2022