

Minimizing Emission for Timely Heavy-Duty Truck Transportation

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Abstract—We consider the problem of minimizing emission of a heavy-duty truck transporting freight between two locations subject to a hard deadline constraint. The truck is equipped with a multi-speed transmission and a modern combustion engine that intelligently switches among multiple fuel injection strategies at certain engine speeds (called *switching speeds*) to achieve lower emission profiles. Our objective is to minimize the emission by optimizing both path and speed planning for heavy-duty trucks with multi-speed transmission and multiple injection strategies in the engine. This emission minimization problem, while pervasive in practice, has two challenges: i) the emission rate function is discontinuous and non-convex due to switching of the fuel injections and gear ratios, which makes the common practice of driving at a constant speed on a road segment not eco-friendly; ii) the problem is NP-hard due to the combinatorial nature of the simultaneous path and speed planning. We tackle the first challenge by considering the case where the truck can travel at a heterogeneous speed profile over a road segment and then formulate the speed planning problem as a convex problem. We further identify special structures in this problem and provide an efficient method for computing the optimal speed profile. We then tackle the second challenge by developing an efficient heuristic for both path planning and speed planning to solve the emission minimization problem on the scale of national highway systems. Our extensive simulations on the US highway system show that our solution reduces up to 46% NOx emission as compared to the commonly-adopted fastest path approach. We also find that optimizing heterogeneous speed profiles reduce up to 32% emission as compared to their homogeneous counterpart, thus are necessary to be considered in eco-friendly truck operations.

Index Terms—Energy-efficient transportation, timely transportation, engine fuel injection strategy, emission

I. INTRODUCTION

In 2021, 11.0 billion tons of freight were transported by heavy-duty trucks in the United States, representing 72.2% of total domestic tonnage shipped [2]. The trucking industry generated \$875.5 billion in revenue in 2021 [2]. This number

Part of this work has been presented at Proceedings of the 7th ACM International Conference on Systems for Energy-Efficient Buildings, Cities, and Transportation (ACM BuildSys), November 18–20, 2020, Virtual Event, Japan [1]. This work is supported in part by a General Research Fund from Research Grants Council, Hong Kong (Project No. 11206821), an InnoHK initiative, The Government of the HKSAR, Laboratory for AI-Powered Financial Technologies, and a Shenzhen-Hong Kong-Macau Science & Technology Project (Category C, Project No. SGDX20220530111203026). The authors would also like to thank the anonymous reviewers for their helpful comments.

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would rank 18th in the world if measured against countries' GDPs. This trend will likely continue, as the global freight activity is predicted to increase by a factor of 2.4 by 2050 [3].

Despite their importance to the economy, heavy-duty vehicles are a significant source of emissions, including Carbon Dioxide (CO₂), Nitrogen Oxides (NO_x), and fine particle matter (PM 2.5). With only 4% of the total vehicle population, heavy-duty trucks produce more than one-third of the CO₂ (the primary greenhouse gas causing global warming) emitted in the transportation sector around the world [3]. In the US, about 16-18% of NO_x is emitted by heavy-duty trucks [4]. In California, one of the most polluted areas in the US, heavy-duty trucks contribute to over 70% of the NO_x emissions from on-road vehicles [5]. Therefore, it is critical to reduce their exhaust emissions for a cleaner environment.

A recent effort to reduce emissions in the transportation sector is the introduction of multiple injection strategies for internal combustion engines, which also improves fuel economy and reduces combustion noise [8]. More specifically, in each engine revolution, the engine control system determines the timing and amount of fuel for several possible injections: (i) the main injection, which provides the bulk of the fuel; (ii) several optional injections (called pilot injections or pre-injections) before the main injection to heat the combustion chamber and ensure a more uniform fuel-air mixture; (iii) one or two optional injections (called post-injections) after the main injection to burn the residual and decrease the amount of pollutants. Fig. 1a illustrates the emission of NO_x as a function of the engine speed for different injection strategies [6], where the emission rate function for each strategy is approximated by a convex function. In addition, these functions are monotonically decreasing with the number of injections: for any engine speed, a higher number of injections means less emission.

However, multiple injections come with the cost of a higher computational load. In addition, when a truck drives at high speed, the time interval for one engine revolution becomes small. Therefore, the limited computational resources and tight real-time requirements may not allow sophisticated control strategies (and multiple injections). Engine control systems are thus often designed to be *self-adaptive* in that they switch to simplified control strategies (e.g., single-injection) at high engine speeds [7]. Fig. 1b shows a typical engine control software realized as a sequence of `if` statements [6], [7]. The control strategy at a speed higher than ω_4 (i.e., $\omega > \omega_4$) only executes function `f1()` for single-injection, compared to (`f1()`–`f5()`) for quintuple-injection at ω_1 . Such engine control software, in general, makes the cost model discontinuous at the switching speeds (thus non-convex). The thick black

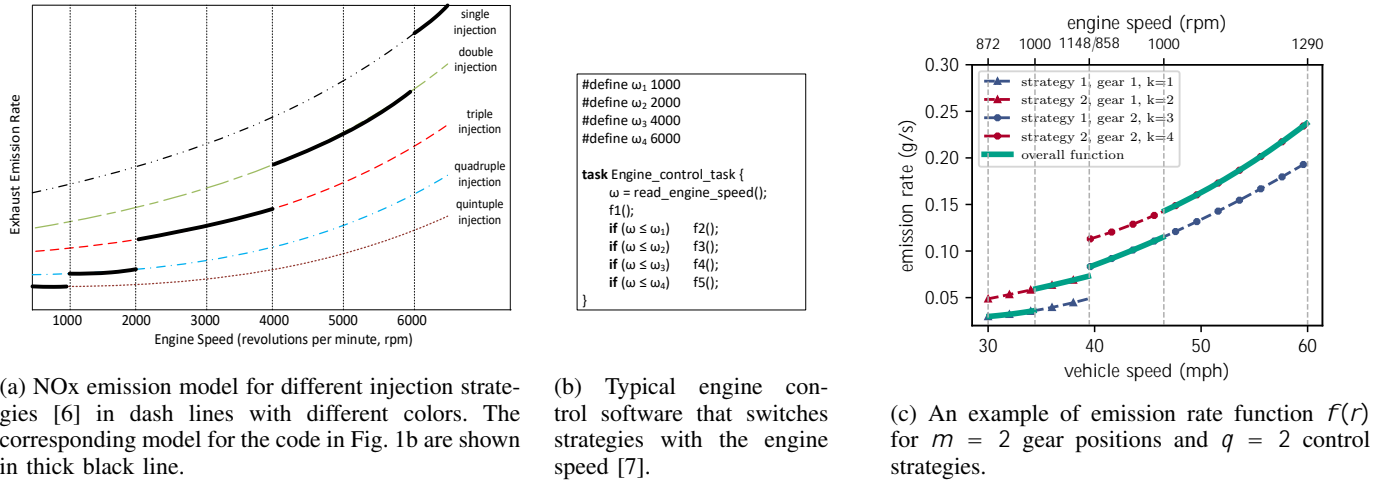


Fig. 1: Illustrations for the discontinuity of the emission rate function.

line in Fig. 1a gives an example of the overall NOx emission model corresponding to the code in Fig. 1b.

Another source of discontinuity in the emission model is gear shifting, i.e., switching the gear so that the engine speed is within the fuel-economic range (see Fig. 2). When the gear is shifted, e.g., from position 9 to 10 at 40 mph, the engine speed falls from 1150 rpm to 850 rpm, which leads to a large difference of engine performance (e.g., emission rate) near the vehicle speed 40 mph. Therefore, the gear-shifting strategy also introduces points of discontinuity to the emission rate function with respect to vehicle speed. An example of the overall emission rate function is presented in Fig. 1c.

Timely transportation is a common requirement in the trucking industry, for three reasons [9]: (i) the nature of the goods (such as fresh food) [10]; (ii) service-level agreements to guarantee delivery delay such as those in Amazon¹, uShip² and Uber Freight³; and (iii) ease of scheduling and operation in the logistics [11]. For instance, mobile applications like uShip and Uber Freight collect freight transportation requests for truck operators, often associated with pickup and delivery time requirements.

In this work, we consider a common truck operation scenario where a long-haul truck drives across a national highway system. Our objective is to minimize the total emission subject to a hard deadline constraint by optimizing path planning and speed planning and leveraging adaptive engine control strategies and multi-speed transmission.

Our problem optimizes path planning and speed planning as in several previous studies [9], [15], [16]. However, we also consider adaptive engine control strategies and multi-speed transmission, which makes our problem uniquely challenging, since the overall emission is a discontinuous and non-convex function with respect to the vehicle speed due to the switching among these control strategies and gear positions at runtime. Table I summarizes the comparison between our work and

related studies. To our best knowledge, we are the first to study the problem of optimizing the operation of a long-haul heavy-duty truck subject to a hard deadline constraint, where the truck is equipped with multiple injection strategies and multi-speed transmission. The **contributions** are summarized as follows.

⊞ We prove that our problem is NP-hard and show that switching among multiple engine control strategies and gear ratios makes the emission rate function discontinuous and non-convex, hence imposing a unique challenge compared to existing studies, which all deal with convex cost functions.

⊞ We find that driving at a constant speed on one road segment is not optimal, and it is necessary to consider *heterogeneous speed profile* at one road segment. We explore the structures of this speed planning problem and formulate it into a convex programming form. We then further leverage the properties of the emission rate functions and derive an efficient method for the optimal speed profile that achieves three orders of magnitude runtime improvement than the convex program solver. We develop an efficient dual-based heuristic for both path planning and speed planning. Moreover, we give a sufficient condition under which the solution of our approach is optimal and an upper bound of the optimality gap when the condition is not satisfied.

⊞ We conduct extensive simulations over the US highway network with 1,000 origin-destination pairs. The results show that our scheme reduces up to 46% emissions compared to the fastest path. We also find that heterogeneous speed profiles reduce up to 32% emission as compared to its homogeneous counterpart, thus are necessary to be considered in environmentally friendly truck operations.

II. RELATED WORK

Restricted Shortest Path (RSP). In an RSP problem, the cost and travel time associated with each road segment are both *fixed*. The objective is to find a path with the minimum total cost such that the total travel time is within a deadline constraint. However, existing approaches and results for RSP

¹Place an Order with Guaranteed Delivery, Amazon, <http://amazon.com>

²uShip, <https://www.uship.com/>

³Uber Freight, <https://freight.uber.com/>

Work	Path Planning	Speed Planning	Hard Deadline	Adaptive Fuel Injection	Transmission	Vehicle Type	Cost	Cost Model
RSP [12], [13], [14]	✓	✗	✓	✗	✗	Truck	Any	Constant
PASO [9], [15], [16]	✓	✓	✓	✗	✗	Truck	Fuel	Convex
Others, e.g., [17], [18], [19]	✗	✓	✗	✗	✓	Truck	Fuel	Nonlinear
Other, e.g., [20]	✗	✓	✓	✗	✓	Truck	Fuel	Nonlinear
Other, e.g., [21]	✓	✗	✗	✗	✗	Truck	Fuel	Constant
Others, e.g., [22], [23]	✗	✓	✗	✗	✓	General	Fuel	Nonlinear
Other, e.g., [24]	✗	✓	✓	✗	✓	General	Fuel	Nonlinear
Other, e.g., [25]	✓	✗	✗	✗	✓	General	Emission	Constant
VRP [26], [27]	✓	✓	✓	✗	✗	General	Emission	Nonlinear
This work	✓	✓	✓	✓	✓	Truck	Emission	Discontinuous

TABLE I: Comparison of our work and existing studies.

are not applicable to our problem, as the fixed cost and travel time for each road segment in RSP disregard the challenging design space of speed planning.

Path selection and Speed Optimization (PASO) and its extensions. PASO [9] generalizes RSP with the additional design space of speed planning. An FPTAS and a dual-based heuristic have been developed to solve the challenging PASO problem [9]. An extension to a multi-task setting has been studied [15] to fulfill multiple transportation tasks under task pickup and delivery time window constraints. The design space of opportunistic driving [16] has been explored to leverage dynamic traffic conditions, wait for benign traffic conditions and reduce the cost. However, all these studies assume the fuel consumption model is convex, while in our problem the cost function is discontinuous and non-convex due to multiple injection strategies and multiple gear positions.

Other related studies. Reducing emission for vehicles has been studied extensively [26], [28], [25], [27]. Bektaş *et al.* [26] propose an extension of the Vehicle Routing Problem (VRP) called Pollution Routing Problem (PRP) with various optimization objectives including fuel consumption, CO2 emission and travel time. Guo *et al.* [25] collect a number of real-world vehicle data and optimize the objectives of fuel consumption and emissions. Hellström *et al.* [17] embed the cost-speed tradeoff in the objective, a weighted sum of travel time and fuel consumption. They also use look-ahead information such as estimated road grade [29] to control the truck's speed profile under a given path. Boriboonsomsin *et al.* [21] present an eco-routing navigation system that determines the most fuel-economic path. Alam *et al.* [30] observe that improved fuel efficiency can be obtained by maintaining the platoon of trucks throughout a hill, motivating subsequent studies, e.g., [31], [32], [33], which focus on developing control strategies for truck platooning to save fuel.

Meanwhile, a continuous effort has been put into engine emission control from both the industry and the regulatory authorities; see a recent review in [34]. In particular, heavy-duty engines are improving at a much slower pace than light-duty ones [34]. However, all these studies focus on the optimization of the design and operation of various parts in the engine (e.g., fuel system and injection strategy [35], [36], [6], [37], exhaust after-treatment system [38]), and they do not consider the planning of path and/or speed profile. For example, Biondi *et al.* [6] present methods to optimize the switching speeds of multiple injection strategies at design time

against standard driving cycles (i.e., with fixed path and speed profile), while Peng *et al.* [39], [37] propose to adjust the switching speeds at runtime using predicted driving cycles.

We summarize the comparison of our work and related studies in Tab. I. To our best knowledge, we are the first to study the problem of minimizing emissions for a long-haul heavy truck equipped with multiple engine control strategies and multi-speed transmission. Compared to existing studies that also simultaneously optimize path planning and speed planning under a hard deadline constraint [9], [15], [16], the consideration of switching among multiple injection strategies and gear ratios makes our problem uniquely challenging due to the discontinuity and non-convexity of the emission rate function.

III. MODEL AND PROBLEM FORMULATION

We model a national highway network as a directed graph $G = (V, E)$ where an edge $e \in E$ represents a road segment, and a node $v \in V$ represents a point of junction for connected road segments. We denote the distance of an edge $e \in E$ by $D^e > 0$. We also denote the minimum (resp. maximum) speed of an edge e by r_l^e (resp. r_u^e). We assume the driving conditions to be homogeneous on each road segment, i.e., the grade and surface resistance are the same on a road segment (otherwise, we can break it into multiple segments). We consider the scenario where a truck travels from an origin $o \in V$ to a destination $d \in V$ across the highway network G within a hard deadline requirement T .

A. Emission Rate function

For a truck and its i -th engine control strategy, we denote the engine performance function by $\hat{f}_i(\omega, p) : \mathbb{R}^2 \rightarrow \mathbb{R}$ where ω is the engine speed in revolutions per minute (rpm), and p is the output power in kW. The output $\hat{f}_i(\omega, p)$ is the (instantaneous) emission rate (unit: g/s).

For a fixed gear position j , the engine speed is linear with respect to the vehicle speed (see Fig. 2). Therefore, we model the mapping from the vehicle speed r to the engine speed ω

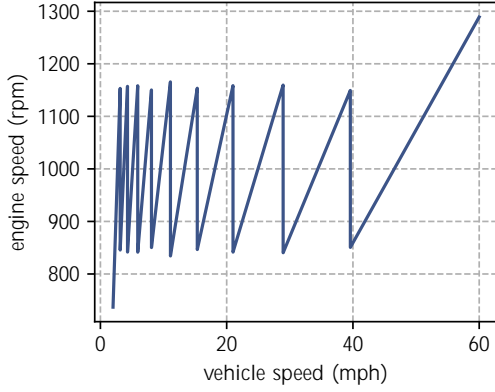


Fig. 2: A typical 10-speed transmission plot from the vehicle speed to the engine speed [40].

as a piece-wise linear function:⁴

$$\omega(r) = a_j r + b_j \quad \text{if } r \in (r_{j-1}^l, r_j^l] \quad (1)$$

for all gear positions $j \in \{1, \dots, mg\}$, where m is the number of gear positions. Moreover, for a truck driving at a constant speed r on a road segment e , we model the output power $p(r)$ as a third-degree polynomial function [43], [44]:

$$p^e(r) = a_3^e r^3 + a_2^e r^2 + a_1^e r + a_0^e \quad (2)$$

Thus the emission rate function of a truck driving at i -th control strategy and j -th gear position over a segment e is

$$f_{ij}^e(r) = \hat{f}_i(a_j r + b_j, a_3^e r^3 + a_2^e r^2 + a_1^e r + a_0^e) \quad (3)$$

The overall emission rate function is given by

$$f^e(r) = f_{ij}^e(r), \text{ if } r \in (r_{j-1}^l, r_j^l] \text{ and } \omega(r) \in (\omega_{i-1}, \omega_i], \quad (4)$$

where ω_i is the switching speed from strategy i to $i + 1$, $i \in \{1, \dots, qg\}$ and q is the number of control strategies. Note that we can write the function in a standard piece-wise form as follows, since the domain for each piece is disjoint

$$f^e(r) = f_k^e(r) = f_{ij}^e(r) \quad \text{if } r \in (s_{k-1}, s_k] \quad (5)$$

for $k \in \{1, \dots, n\}$. Here $n = mg$ is the number of pieces. We set the boundary points to the speed limits on edge e : $s_0 = r_l^e$, $s_n = r_u^e$. We then derive each s_k , $k \in \{1, \dots, n-1\}$ as:

$$s_{k-1} = \frac{\omega_{i-1} b_j}{a_j}, \quad s_k = \frac{\omega_i b_j}{a_j}$$

$$i = k - q \frac{k-1}{q}, \quad j = \frac{k-1}{q} + 1$$

Fig. 1c gives an illustrative example of the emission rate function $f^e(r)$ with $n = 4$ pieces. There are three points of discontinuity in total: two of them are due to switching of

⁴ In practice, drivers often shift the gear position to keep the engine speed within a ‘‘sweet spot’’ range [41], [42] to ensure smooth driving and engine performance. In this study, we consider that the driver follows such a gear-shifting strategy during the whole trip. Consequently, this paper does not optimize the vehicle-to-engine speed map. That is, given the vehicle speed r , the engine speed is determined by a fixed piece-wise linear function $\omega(r)$ like Fig. 2.

control strategies at 1,000 rpm and one of them is due to switching of gear positions at 40 mph.

In this paper we make the following assumptions on the high-level emission rate $f^e(r)$:

piece-wise convex: each function $f_{k_1}^e$ is convex over the interval $[r_{l_1}^e, r_{u_1}^e]$;

staircase-shaped: they satisfy that

$$\delta k_1 < k_2 \in [n], \delta r \in [r_{l_1}^e, r_{u_1}^e], \quad f_{k_1}^e(r) < f_{k_2}^e(r) \quad (6)$$

where $[n]$ is the set of positive integers no larger than n .

We remark that the above assumptions on the cost model are realistic for a number of metrics related to eco-friendly truck operations. For example, the fuel rate can be described as an increasing polynomial function of the vehicle speed, verified with both theoretical and empirical studies [9]. The amount of CO₂ emission is roughly proportional to the amount of fuel consumption (e.g., about 10.18kg per gallon of diesel, or 8.887kg per gallon of gasoline) [45]. Hence it exhibits the same characteristics as the fuel-rate function. Similarly, the NO_x emission can be modeled as an exponential function that remains convex in the engine speed range based on extensive simulation data [6]. The rationale behind the second assumption (6) is that a more complicated and time-consuming fuel injection strategy makes sense only if it provides some benefit such as reduced emission, but its complexity makes it only feasible at lower speeds [6].

While our approach generally applies to any cost function that satisfies the above assumptions, we focus on minimizing the emission cost for concreteness in the rest of this paper. Those assumptions are also verified later in our simulations.

B. Total Emission Function

In contrast to convex cost rate functions discussed in existing literature [9], [15], [16], the emission rate function $f^e(r)$ we study is discontinuous and non-convex which makes it non-trivial to optimize. In particular, while it is optimal to drive at a constant speed to travel through one road segment with a convex cost rate function [9], this approach is sub-optimal for discontinuous cost functions. In such cases, operating within two speed ranges can yield lower costs, as demonstrated in Example 1 in Appendix VIII-A. Therefore, it is necessary to consider a *heterogeneous speed profile* on one road segment to minimize emission costs. In the following, we shall study how to optimize the heterogeneous speed profile and compute the total emission function $c^e(t)$ that gives the minimum emission cost (in grams) for a truck to traverse the edge e with the travel time of t . We first present a similar result to [9, Lemma 1].

Lemma 1. *If the total travel time t_i following the speed in the range (s_{i-1}, s_i) of the i -th piece $f_i(r)$ is given, then the optimal speed profile is to maintain some constant speed r_i for the whole duration t_i .*

Proof. Similar to [9, Lemma 1], it is proven by applying the continuous Jensen’s inequality to the convex function $f_i(r)$. \square

Given Lemma 1, we are now ready to formulate the problem of optimizing the speed profile to pass an edge e , with the

length of the edge D^e , and the total travel time t^e . We denote r_k and t_k as the selected speed and travel time for the k -th piece, respectively. We introduce the auxiliary variable $d_k = r_k t_k$ to denote the driving distance for the k -th piece and formulate the problem as follows⁵

$$c^e(t^e) = \min_{t_k} \sum_{k \in \mathcal{X}} t_k f_k \frac{d_k}{t_k} \quad (7a)$$

$$\text{s.t.} \quad \sum_{k \in \mathcal{X}} t_k = t^e, \quad \sum_{k \in \mathcal{X}} d_k = D^e \quad (7b)$$

$$s_k \leq t_k \leq d_k, \quad \forall k \in \mathcal{X} \quad (7c)$$

The objective (7a) is to minimize the total emission on the road segment. The constraint (7b) makes sure that the total driving distance is equal to the length D^e of the road segment and that the total travel time is equal to t^e . The constraint (7c) is to make sure the driving speed $\frac{d_k}{t_k}$ is within the corresponding speed range for each piece k .

Lemma 2. *The speed planning problem in (7) is a convex optimization problem.*

Proof. In the objective, each summand $t_k f_k \frac{d_k}{t_k}$ is the perspective of the convex function f_k , thus is also convex [47] in d_k and t_k . Therefore, the objective function is the sum of convex functions and thus is convex. Then convexity of the objective combined with the observation that all the constraints (7b)-(7c) are linear, giving the desired result. \square

Therefore, we can efficiently compute the total emission function $c^e(t^e)$ by solving problem (7) as a standard convex program. This approach, however, only exploits the piece-wise convexity of the emission rate function $f^e(r)$. In Section IV, we will delve deeper into the unique structures of this problem and introduce a more efficient method. Notably, the proposed method offers a runtime that is three orders of magnitude faster than directly solving the convex program (7), as detailed in Table III in Section VI.

C. Problem Formulation

In this paper, we consider the problem of minimizing the total emission cost for a truck to travel from an origin o to a destination d within a hard deadline T . We consider the problem from the driver's perspective, where the truck parameters and engine parameters are predefined. The problem inputs include the graph $G = (V, E)$, the emission rate function $f^e(r)$ for each road segment e that incorporates the truck's and its engine's characteristics, the vehicle speed to engine speed map $\omega(r)$, the speed limits r_l^e and r_u^e for each edge e , the origin o , the destination d , and the deadline T .

The solution to our problem include a path from o to d and the vehicle speed(s) on each road segment along the path. The

⁵In this paper, we ignore the emission cost due to acceleration and deceleration during the speed transition phase, because this phase usually spans over only several hundred feet [46] while the length of one road segment is several miles or longer. Meanwhile, as we will show in Lemma 3, it is sufficient to drive with at most two different speeds to achieve the minimum emission cost at one road segment and our method suggests to travel for more than a mile every speed change on average (see Fig. 6).

constraints include the speed limits on each road segment, and a hard deadline that requires the total travel time to be no more than T . In particular, we formulate our problem as follows:

$$\min_{\mathcal{X}} \sum_{e \in \mathcal{X}} x^e c^e(t^e) \quad (8a)$$

$$\text{s.t.} \quad \sum_{e \in \mathcal{X}} x^e t^e \leq T, \quad (8b)$$

where \mathcal{X} defines a simple path from o to d

$$\mathcal{X} = \left\{ x : x^e \geq 0, \forall e \in E, \text{ and } \sum_{e \in \text{out}(v)} x^e = \sum_{e \in \text{in}(v)} x^e = 1_{v=o}, 1_{v=d}, \forall v \in V \right\}.$$

Here 1_{FG} is the indicator function, $\text{in}(v) = \{e \in E : (u, v) \in e\}$ is the set of incoming edges of node v , $\text{out}(v) = \{e \in E : (v, u) \in e\}$ is the set of outgoing edges of node v . The set T captures the speed limits of all roads, which is defined as

$$T = \{t_l^e \leq t^e \leq t_u^e, \forall e \in E\},$$

where $t_l^e = \frac{D^e}{r_u^e}$ and $t_u^e = \frac{D^e}{r_l^e}$ are the minimum and maximum travel times of traversing the edge e , respectively.

We remark that there are two challenges to solve the problem (8). The first challenge comes from the discontinuity and the non-convexity of the emission rate function $f^e(r)$, which makes it more difficult to compute $c^e(t)$ than previous work [9], [15], [16]. The second challenge comes from the combinatorial nature of the problem 8, which makes it NP-hard by the following theorem.

Theorem 1. *The problem (8) is NP-hard.*

Proof. This directly follows the fact that the NP-complete problem PASO [9] is a special case of our problem, where PASO only considers one control strategy (i.e., $n = 1$). \square

In the following two sections, we shall tackle those two challenges. We shall provide an efficient method for solving the speed planning problem (7) by identifying special structures of the problem in Sec. IV. In Sec. V, we shall provide an efficient dual based method that solves the overall path planning and speed planning problem (8).

IV. SPEED PLANNING

In this section, we shall present an efficient method for computing $c^e(t)$ by exploring the properties of the emission rate function $f^e(r)$. For ease of presentation, we omit the superscript e from the notations in the rest of the section, e.g.,

(a) Illustration for Lemma 4. Shifting the left point $(r_1; y_1)$ to right point $(r_3; y_3)$ decreases the cost. (b) Illustration for Lemma 5 for small. The objective is minimized at the generalized right point of tangency. (c) Plot for $f^*(r)$ in solid line. The dotted lines are functions $f_1(r_1)$ and $f_2(r_2)$ respectively.

Fig. 3: Illustrations for ideas of solving the speed planning problem.

$f^e(r)$ is simplified as $f(r)$. We now rewrite problem (7) in the following equivalent form to better illustrate our method following lemma.

$$f(r) = \min_{i \in \{0, r_i \in [0, 8i^2/n]\}} \sum_{i=1}^n f_i(r_i) \quad (9a)$$

$$\text{s.t.} \quad \sum_{i=1}^n r_i = r \quad (9b)$$

$$r_i \in [s_{i-1}, s_i]; \quad i \in [1, n] \quad (9c)$$

$$r_i \in [s_{i-1}, s_i]; \quad i \in [1, n] \quad (9d)$$

where $r_i = \frac{t_i}{t}$, and $r = \frac{D}{t}$. Essentially, problem (9) is to find a set of speeds, one for each piece, such that their convex combination is equal to the average speed $\frac{D}{t}$ over the edge, and the weighted average of the emission rate is minimized (denoted as $f^*(r)$). Note that this reformulation is not convex anymore due to the nonlinear equality constraint (9b) and the non-convex objective. However, problem (9) has special structures that allow us to design an efficient method for solving it. The proposed method is three orders of magnitude faster than directly solving the convex program (7) (cf. Tab. III in Sec. VI). In the following, we shall present this method for the function f with $n > 1$ pieces, because when $n = 1$, the optimal solution is simply $r_1 = r$ and $r_1 = 1$ by Lemma 1. We outline the derivation steps as follows:

We first show by Lemma 3 that we only need to focus on at most two out of n pieces of function f to obtain the optimal solution.

Given i -th piece and j -th piece of function f_i and f_j with $1 \leq i < j \leq n$, we show by Lemma 4 that how to choose the speed for i -th piece and show by Lemma 5 that how to choose the speed for j -th piece. Therefore, we can obtain the minimum-cost solution on two specified pieces $f_i; f_j$.

Finally, we present our efficient method by searching over $O(n^2)$ pairs of candidate pieces and identifying the optimal solution.

Next, we present the details on how to derive our efficient method for solving the speed planning problem (9). Suppose the average speeds in the i -th piece of the function f , i.e., $r \in [s_{i-1}, s_i]$. Let $r^? = [r_1^?; r_n^?]$ and $r^? = [r_1^?; r_n^?]$

Lemma 3. There exists an optimal solution $r^?$ that has at most two non-zero entries.

Proof. See Appendix VIII-B. \square

Lemma 3 suggests that we can focus on finding only two points on the curve of the function whose convex combination is minimized. Fig. 3a gives a geometric illustration of our goal. We want to find two points $(r_1; y_1)$ and $(r_2; y_2)$ on two pieces of function f such that the interpolated point $(r; y_2)$ has the minimum function value y_2 . To investigate how to choose two points on two given pieces. We first fix the right point $(r_2; y_2)$ and shift the left point $(r_1; y_1)$ to reduce the cost, which yields the following lemma.

Lemma 4 (Illustrated by Fig. 3a) Consider a convex function $g(r)$, a fixed point $(r_2; y_2)$ above the curve of $y = g(x)$, i.e., $y_2 > g(r_2)$. For every $r_1 < r$, let $(r_1; y_1)$ be the convex combination of two points $(r_1; y_1 = g(x_1))$ and $(r_2; y_2)$, i.e.,

$$y = \frac{y_2 - y_1}{r_2 - r_1} (r - r_1) + y_1 \quad (10)$$

Then for every fixed $r_2 \leq r$, y is monotonically non-increasing with r_1 for $r_1 < r$. In addition, $y \geq g(r)$.

Proof. See Appendix VIII-C. \square

Fig. 3a illustrates the geometric interpretation of Lemma 4. It suggests that given fixed right point $(r_2; y_2)$, we should always choose larger point $(r_3; y_3)$ on the right boundary of the same piece to decrease the objective. Next, we consider the case where the left point $(r_1; y_1)$ is fixed and we want to shift the right point $(r_2; y_2)$ to reduce the cost, which is summarized in the following lemma.

Lemma 5 (Illustrated by Fig. 3b) Consider a convex function $g(r)$ and a fixed point $(r_1; y_1)$ that is below the curve of $y = g(r)$, i.e., it satisfies $y_1 < g(r_1)$. Define $(p; y_p = g(p))$ as the generalized right point of tangency from $(r_1; y_1)$ to its right hand side of the curve $y = g(r)$, i.e., $p > r_1$ satisfies the following equation

$$\frac{g(p) - y_1}{p - r_1} = g'(p) \quad (11)$$

where $g'_-(p)$ and $g'_+(p)$ are the left and right derivatives of g at p .

Given $r > r_1$, for every $r_2 \leq r$, let $(r; y)$ be the convex combination between $(r_1; y_1)$ and $(r_2; y_2 = g(r_2))$, that is,

$$y = \frac{y_2 - y_1}{r_2 - r_1} (r - r_1) + y_1 \quad (12)$$

Then y has the following properties:

If $r \leq p$, then y is monotonically non-increasing with r_2 until p , and monotonically non-decreasing with r_2 afterwards. In addition $g'_-(r_2) \leq y \leq g'_+(r)$.

If $r > p$, then y is monotonically non-decreasing with r_2 . In addition, $g'_-(r_2) \leq y \leq g'_+(r)$.

Proof. See Appendix VIII-D □

Fig. 3b illustrates the geometric interpretation of Lemma 5 when n is small. It suggests that given fixed left point $(r_1; y_1)$, we want the right point $(r_2; y_2)$ as close to $(r_p; y_p)$ as possible to reduce to cost. Meanwhile, when n is larger than a threshold p , we can simply drive at the constant speed v .

The efficient method. With Lemma 4 and Lemma 5, we can directly obtain the optimal solution if the optimal piece(s) are known. Combined with Lemma 3, we notice that it is sufficient to check $O(n^2)$ pairs of points to obtain the optimal solution. In particular, we summarize our efficient method in the following:

Theorem 2. For any two pieces i, j such that $1 \leq i < i_r \leq j \leq n$, we denote by p_{ij} the generalized right point of tangency from point $(s_i; f_i(s_i))$ to the curve $y = f_j(r)$. We also denote the clamped p_{ij} as

$$p_{ij} = \begin{cases} s_j - 1; & \text{if } p_{ij} < s_j - 1; \\ p_{ij}; & \text{if } p_{ij} \in [s_j - 1; s_j]; \\ s_j; & \text{if } p_{ij} > s_j \end{cases} \quad (13)$$

Then by Lemma 4 and Lemma 5, the minimum objective and the corresponding solution on piece i, j are given by:

$$\begin{aligned} r_i &= s_i; \quad r_j = p_{ij} \\ i &= \frac{r - r_i}{r_j - r_i}; \quad j = \frac{r_j - r}{r_j - r_i} \\ y_{ij} &= i f_i(s_i) + j f_j(p_{ij}) \end{aligned}$$

We additionally define the case $i = j = i_r$:

$$r_{i_r} = r; \quad i_r = 1; \quad y_{i_r, i_r} = f_{i_r}(r)$$

Then the optimal solution and objective to the problem (9) are identified by:

$$\begin{aligned} (i^*; j^*) &= \arg \max_{1 \leq i, j \leq n} y_{ij} \\ y^* &= \max_{1 \leq i, j \leq n} y_{ij} \end{aligned}$$

Proof. The proof directly follows from combining Lemma 3, Lemma 4, and Lemma 5. □

Therefore, we only need to check $(n - i_r + 1) \cdot n^2$ number of candidate points to solve the problem (9). There are other techniques to reduce the number of candidates by further exploring the relationship between different pairs of pieces.

However, we remark that the proposed method is efficient enough in practice because the number of pieces is relatively small. In fact, since we are considering a truck travelling across the highway system with vehicle speed larger than 120 mph on most road segments, there are only $m = 3$ gear positions involved in our scenario (cf. Fig. 2). If we consider an engine with $q = 3$ control strategies, then the number of pieces $n = mq = 9$ is indeed small. Therefore, Theorem 2 provides an efficient method for computing $c(t) = t \cdot f'(D=t)$. It allows us to efficiently solve the overall emission minimization problem (8) in the next section. Fig. 3c gives an illustration of $f'(r)$. It can be seen as the continuous version of the original discontinuous function $f(r)$ by drawing a line between the break point $(s_1; f_1(s_1))$ and $(r_p; y_p)$. As one may observe from Fig. 3c, we can show that $f'(r)$ is convex with respect to r and thus $c(t) = t \cdot f'(D=t)$ is also convex, as stated in the following proposition.

Proposition 1. $c(t)$ is convex over $[t_l; t_u]$.

Proof. See Appendix VIII-E. □

In the next section, we shall leverage the efficient computation method and convexity of $c(t)$ to provide an efficient dual-based method that solves the overall emission minimization problem (8).

V. AN EFFICIENT DUAL-SUBGRADIENT ALGORITHM WITH PERFORMANCE GUARANTEE

With the efficient method to the speed planning for a given travel time t on an edge and hence a fast computation for $c^e(t)$, we consider the overall path planning and speed planning problem (8) which now amounts to finding a path and assign a travel time to each edge on the path, such that the emission is minimized and the subject to the hard deadline constraint T . We shall design an efficient heuristic based on Lagrangian relaxation and derive a theoretical condition under which our heuristic outputs the optimal solution. As illustrated in the simulations, our heuristic quickly finds close-to-optimal solutions for the scale of the US national highway network.

A. Lagrangian Relaxation and Dual Problem

We introduce a Lagrangian dual variable θ , and derive the Lagrangian relaxation for problem (8) as

$$\begin{aligned} L(x; t; \theta) &= \sum_{e \in E} x^e c^e(t^e) + \theta \left(\sum_{e \in E} x^e t^e - T \right) \\ &= \sum_{e \in E} x^e (c^e(t^e) + \theta t^e) - \theta T \end{aligned}$$

The corresponding dual function is given by

$$D(\theta) = \min_{x \geq 0, t \geq T} L(x; t; \theta)$$

and the dual problem of the original problem (8) is

$$\max_{\theta} D(\theta)$$

Given λ , we can obtain the value $D(\lambda)$ by computing the shortest path with easily computed weights for each edge. In particular, we have

$$\begin{aligned}
 D(\lambda) &= T + \min_{x \in \mathcal{X}} \sum_{e \in E} x^e (c^e(t^e) + t^e) \\
 &= T + \min_{x \in \mathcal{X}} \sum_{e \in E} x^e \min_{t_i^e, t_u^e} (c^e(t^e) + t^e) \\
 &\stackrel{(a)}{=} T + \min_{x \in \mathcal{X}} \sum_{e \in E} x^e (c^e(t^e(\lambda)) + t^e(\lambda)) \\
 &\stackrel{(b)}{=} T + \min_{x \in \mathcal{X}} \sum_{e \in E} x^e w^e(\lambda) \\
 &\stackrel{(c)}{=} T + \sum_{e \in \mathcal{P}(\lambda)} w^e(\lambda); \tag{14}
 \end{aligned}$$

Here $t^e(\lambda)$ in (a) is defined as

$$t^e(\lambda) = \arg \min_{t_i^e, t_u^e} c^e(t^e) + t^e; \tag{15}$$

That is, $t^e(\lambda)$ is the optimal travel time that minimizes $c^e(t^e) + t^e$ for edge $e \in E$, $w^e(\lambda)$ in (b) is the corresponding optimal cost

$$w^e(\lambda) = c^e(t^e(\lambda)) + t^e(\lambda); \tag{16}$$

and $\mathcal{P}(\lambda)$ in (c) is the resulting minimum-cost path where each edge is associated with an edge cost $w^e(\lambda)$. Given a value to the dual variable, Equation (14) suggests that we can figure out $D(\lambda)$, the value of the dual function, by finding a shortest path with each edge assigned an edge cost of $w^e(\lambda)$. In the following, we first derive an analytical solution to $t^e(\lambda)$ and hence $w^e(\lambda)$ for each edge $e \in E$, then in Section V-B we propose an iterative procedure to find an appropriate value for λ .

From Section IV, we can efficiently compute the value of $c^e(t)$ by checking a small number of candidates. Moreover, Proposition 1 shows that $c^e(t)$ is convex (but potentially piece-wise), hence it allows left and right derivatives. The following lemma provides an analytical solution $t^e(\lambda)$ and hence $w^e(\lambda)$.

Lemma 6. In case that $\lambda \in [c^e(t_i^e), c^e(t_u^e)]$, denote t as any t such that $\lambda \in [c^e(t), c^e(t)]$, which is well-defined since the derivatives of the convex function $c^e(t)$ are non-decreasing. Then $t^e(\lambda)$ is given as

$$t^e(\lambda) = \begin{cases} < t_i^e; & \text{if } \lambda + \lambda' c^e(t_i^e) > 0 \\ t; & \text{if } \lambda \in [c^e(t_i^e), c^e(t_u^e)] \\ t_u^e; & \text{if } \lambda + \lambda' c^e(t_u^e) < 0 \end{cases} \tag{17}$$

Proof. Observe that $c^e(t) + t$ is also convex with respect to t . Hence its derivatives are non-decreasing.

If $\lambda + \lambda' c^e(t_i^e) > 0$, then $c^e(t) + t$ is non-decreasing for $t < t_i^e$, hence its minimum is achieved at the lower bound of t , i.e., t_i^e . If $\lambda + \lambda' c^e(t_u^e) < 0$, then $c^e(t) + t$ is non-increasing for $t > t_u^e$, hence its minimum is achieved at the upper bound of t , i.e., t_u^e .

If $\lambda \in [c^e(t_i^e), c^e(t_u^e)]$, then the derivatives of $c^e(t) + t$ remain to be non-positive for $t < t$, and then are

When $\lambda \in [c^e(t_i^e), c^e(t_u^e)]$, we can find t with a binary search scheme, because the derivatives of the convex function c^e is non-decreasing. The complexity is thus $O(n^2 \log \frac{t_u^e - t_i^e}{\epsilon})$ where n is the number of control strategies and ϵ is the error tolerance for.

B. Our Heuristic Algorithm

For a given λ , we define the total travel time of the minimum-cost path $\mathcal{P}(\lambda)$ as

$$P(\lambda) = \sum_{e \in \mathcal{P}(\lambda)} w^e(\lambda); \tag{18}$$

We introduce an important observation of $P(\lambda)$ below.

Lemma 7. $P(\lambda)$ is non-increasing over $\lambda \in [0, +\infty)$.

Proof. Refer to [9, Thm. 3], which is still applicable to our problem since it only uses the facts that $\mathcal{P}(\lambda)$ minimizes $c^e(t) + t$ and $\mathcal{P}(\lambda)$ is the minimum-cost path. \square

By the Lagrangian dual relaxation, the value $D(\lambda)$ as calculated in (14) is always a lower bound of the minimized emission to the original problem (8). Hence, we observe that the Lagrangian dual variable with $\lambda = T$ defines the

optimal solution \mathcal{P}^* . By Lemma 7, our heuristic suggests to use a binary-search scheme to update λ approach, by comparing $P(\lambda)$ with T . The details of our heuristic are described in Algorithm 1, where ϵ is the error tolerance for finding a shortest path with each edge assigned an edge cost. λ can be interpreted as a price on the delay. Hence λ can be set as the upper bound on the emission rate.

In the algorithm, whenever we find a λ such that $P(\lambda) = T$ (Line 8), it must be the optimal value. If $P(\lambda) > T$ (Line 10), then the deadline constraint is violated, and we set λ as the new lower bound. Otherwise (Line 12), we set λ as the new upper bound, and update the current best solution \mathcal{P} and t .

Algorithm 1 Our Heuristic Approach

```

1: procedure
2:   Set  $\lambda_l = 0$  and  $\lambda_u = \max$ 
3:   while  $\lambda_u - \lambda_l > \epsilon$  do
4:     Set  $\lambda = \frac{\lambda_l + \lambda_u}{2}$ 
5:     Obtain  $t^e(\lambda)$  according to Lemma 6 for all  $e \in E$ 
6:     Set  $w^e(\lambda)$  according to Equation (16) for all  $e \in E$ 
7:     Get the shortest path  $\mathcal{P}(\lambda)$  from  $o$  to  $d$  in  $G$ 
8:     if  $P(\lambda) = T$  then
9:       return  $\mathcal{P}(\lambda)$  and  $t^e(\lambda)$ ;  $\forall e \in E$ 
10:    else if  $P(\lambda) > T$  then
11:      Set  $\lambda_l = \lambda$ 
12:    else
13:      Set  $\lambda_u = \lambda$ ,  $\mathcal{P} = \mathcal{P}(\lambda)$ , and  $t^e = t^e(\lambda)$ ;  $\forall e \in E$ 
14:    return  $\mathcal{P}$  and  $t^e$ ;  $\forall e \in E$ 

```

We remark that our heuristic has a strong theoretical performance guarantee. That is, Algorithm 1 always returns a

feasible solution as long as the problem is feasible. Moreover, we give a sufficient condition under which the solution of our approach is optimal and an upper bound of the optimality gap when the condition is not satisfied.

Theorem 3. If Algorithm 1 returns in Line 9, then the returned solution is optimal to our problem. Otherwise if Algorithm 1 returns in Line 14, the returned solution $\mathbf{S} = (p^* \text{ and } t^e; 8e2 \text{ Eg})$ satisfies the deadline constraint and hence is feasible. Furthermore, it has the following theoretical performance guarantee:

$$C(\mathbf{S}) \leq \text{OPT} + \epsilon(T); \quad (19)$$

where $C(\mathbf{S})$ is the total emission of the solution \mathbf{S} , OPT is the optimal emission of our problem, and ϵ is the value of the dual variable corresponding to the returned solution.

Proof. By Equation (14), for a given dual variable, the duality gap of our problem is:

$$\text{Duality Gap} = \epsilon(T); \quad (20)$$

hence the theorem holds. \square

Time Complexity. We now analyze the time complexity of Algorithm 1. The total number of iterations is $O(\log \frac{t_{\max}}{t_{\min}})$. Within each iteration, the calculation of the optimal μ and $w^e(\cdot)$ for all edges takes time $O(n^2 M \log \frac{t_{\max}}{t_{\min}})$ where M is the number of edges in the highway network, n is the total number of control strategies of the engine, t_{\max} (resp. t_{\min}) is the maximum (resp. minimum) travel time among all edges. The step of finding the shortest path takes time $O(M + N \log N)$, where N is the number of vertices. Overall, Algorithm 1 has a time complexity of $O(\log \frac{t_{\max}}{t_{\min}} \cdot n^2 M \log \frac{t_{\max}}{t_{\min}} + M + N \log N)$.

VI. PERFORMANCE EVALUATION

In this section, we present simulation results with real world traces to evaluate the performance of our algorithm. Our objectives are i) to study the impact of heterogeneous speed over one road segment; ii) to study the impact of the deadline on the performance of different approaches. iii) to study the performance of the proposed approach as compared to the conceivable alternatives;

A. Simulation Setup

Transportation Network. We collect the highway network data from the Map-based Educational Tools for Algorithms Learning (METAL) project [48]. The constructed network consists of 84504 nodes and 178238 directed edges. The grade of each road segment is derived from the elevations of its end nodes provided by the Shuttle Radar Topography Mission (SRTM) [49] project.

Traffic Data. We set the maximum speed v_{\max}^e of a road segment as the historical average speed by collecting real-time speed data from HERE map [50] for two weeks. The minimum speed v_{\min}^e is manually set to be $v_{\min}^e = \min\{30 \text{ mph}, v_{\min}^e\}$.

Origin	Destination	Distance (miles)	Value (billion USD)
Los Angeles CA	Columbus OH	1977	17.725
Los Angeles CA	Dallas-Fort Worth TX	1240	12.247
Los Angeles CA	Chicago IL	1745	11.293
Los Angeles CA	Nashville TN	1780	10.718
Los Angeles CA	Houston TX	1373	7.837

TABLE II: Five popular origin-destination pairs from Los Angeles. Here the distance means the straight-line distance between the origin and destination.

Fig. 4: The fitted emission rate function with respect to the vehicle speed with switching speed $\omega = 1000$ rpm.

Origin-destination pair. We collect origin-destination pairs from the Freight Analysis Framework (FAF) [51]. We select 1,000 origin-destination pairs with distances longer than 1,000 miles. Those pairs represented 950 billion dollars of freight by trucks in 2017. Tab. II illustrates a subset of the selected origin-destination pairs in the US starting from Los Angeles.

Emission Model. We consider a class-8 heavy-duty truck Kenworth T800 with a 36-ton full load [52]. We collect the emission data for a typical four-stroke six-cylinder diesel engine [41] using the engine simulation software Diesel-RK [53]. We consider two fuel injection strategies: the traditional single injection and the cost-effective triple injection. We use the 10-speed transmission data from [40]. We consider the case where the driver follows the gear-shifting strategy that maintains the engine speed at around 1,000 RPM. The resulting engine speed to vehicle speed mapping is illustrated in Fig. 2. We then fit each piece of fuel rate function by a three-order polynomial. As shown in Fig. 4, the overall emission rate function satisfies the two assumptions in Sec. III.

Baseline Comparison. In the simulations, we implement our algorithm in Julia [54]. We run all the simulations on a desktop with 13th Gen Intel(R) Core(TM) i5-13400F processor and 64 GB RAM. We implement and compare the following approaches:

- FAST:** The fastest path driving at its maximum speed. Given a graph G , we set the weight of each edge e as its minimum travel time $D^e = r_{\min}^e$. We then find the fastest path from the origin to the destination D using the Dijkstra algorithm.
- MFI:** Our heuristic algorithm with both single-injection

and triple-injection control strategies. By default, we set the switching speed to 1,000 rpm. We set τ and t to be both 10^{-6} .

MFI-NS: Same as MFI but with a constant speed over one road segment. That is, we use $u^e(t) = t^e f^e(D^e = t^e)$ in Algorithm 1. We use the golden-section search method to obtain $t^e(\cdot)$ in (15).

SFI: Our heuristic algorithm with only the single-injection strategy. That is, the emission rate function $f^e(r)$ only reflects the single-injection strategy. Note that the $f^e(r)$ is still non-convex due to different gear positions at different driving speeds.

TFI: Our heuristic algorithm with only the triple injection strategy. That is, the emission rate function $f^e(r)$ only reflects the triple-injection strategy. This approach gives a performance upper bound to MFI, but it is not computationally feasible for the engine's software due to the high load of the triple-injection strategy.

Deadline Selection and Performance Evaluation Given the origin and destination, we denote T_f as the minimum traveling time by FAST. We then set the deadline $T = T_f$, where the τ ranges from 1:05 to 2:0 in the simulation. We call τ the delay factor. For two algorithms A_1 and A_2 , with emission costs C_1 and C_2 , the relative emission reduction of A_1 as compared to A_2 is given by $(C_2 - C_1)/C_2 = 100\%$.

B. A Case Study

We first present a case study from Los Angeles, CA, to Columbus, OH, which represents the first row in Tab. II. We set the switching speed to 1,000 rpm and the deadline to be $T = 1.5T_f$ to better illustrate the benefit of heterogeneous speed planning. Using our MFI approach, we determine an emission-efficient solution that includes both a path and a speed profile, depicted by the blue lines in Figure 5. For comparison, we also present a modified speed profile that remains constant over each road segment, shown by the red lines in the same figure. To provide a clearer view, we zoom into a section of the trip in Figure 5, showcasing the truck's operational states. In the first two road segments, our approach MFI suggests driving at heterogeneous speed profiles, maintaining the engine speed just below the switching speed (1,000 rpm) to leverage the benefit of the triple-injection control strategy. Such a heterogeneous speed profile indeed achieves a smaller emission cost than the homogeneous speed profile, as evidenced in the last row of Figure 5.

To assess computational efficiency, we compare the runtime of calculating the emission function $\phi(t)$ using the convex program (7) and our efficient method introduced in Theorem 2. The convex problem is solved using the nonlinear programming solver MadNLP [55] integrated with JuMP [56]. The results, presented in Table III, reveal that our method is three orders of magnitude faster than directly solving the convex problem. Such improvement makes our method practical for computing the emission-efficient timely transportation plan on the national scale highway network.

Fig. 5: A demonstration of the operational states of a truck traveling from Los Angeles, CA to Columbus, OH, using our MFI approach. The figure zooms into a specific section of the journey for enhanced visibility. The switching speed is 1,000 rpm, and the delay factor is 1.5. Each road segment is delineated by two vertical grey dashed lines.

	Convex program (7)	Our efficient method for problem (9)
Total runtime (seconds)	23,271	11
Number of solves	$5.1 \cdot 10^6$	$5.1 \cdot 10^6$
Average runtime per solve	$4.5 \cdot 10^{-3}$	$2.2 \cdot 10^{-6}$

TABLE III: Runtime comparison of the convex program (7) and our efficient method for problem (9) for solving the case study with MFI.

Fig. 6: Impact of heterogeneous speed over one road segment. The relative emission reduction of MFI as compared to MFI-NS is shown as green line.

Fig. 7: Average relative emission reduction of different algorithms as compared to FAST w.r.t. delay factor. The switching speed is 1,000 rpm.

C. Impact of heterogeneous speed over one road segment

We found in Sec. IV that the driver can drive with heterogeneous speed profile on one road segment to achieve emission reduction. In this subsection, we further explore this interesting observation by s. We set the engine switching speed to 1000rpm and conduct the simulations with different delay factors over 1,000 origin-destination pairs. The results are presented in Fig. 6. When the deadline is tight (e.g., $\tau = 1:1$), the design space of speed planning is small, the heterogeneous speed profile only saves 4% emission as compared to its homogeneous counterpart MFI-NS. As the deadline gets relaxed MFI can leverage more design space from heterogeneous speed planning over one road segment thus can achieve up to 32% emission reduction on average. Therefore, it is necessary to consider heterogeneous speed profiles in environmentally friendly truck operations.

However, this emission reduction comes with side effects. The driver has to change the speed more frequently to achieve this further emission reduction. In particular, when the delay factor is 1:1, our approach MFI suggests changing speed every 1:85 miles on average while MFI-NS suggest changing speed every 2:38 miles. When the τ increases to 2:0, MFI increases the speed change frequency to 4 miles per speed change to further leverage the design space of speed planning.

D. Impact of Deadline

We now study the impact of timely delivery requirements on different approaches. We set the switching speed to 1,000 rpm and vary the delay factor from 1:05 to 2:0. Fig. 7 illustrates the relative emission reductions of different approaches compared to FAST. We observe that as the delay factor increases, all compared approaches reduce more emissions because a relaxed deadline allows for slower travel speeds and more space for speed planning, resulting in more emission reduction. Meanwhile, when the delay factor exceeds 2:0,

the travel speeds reach the lower bounds, and there is no further space for speed planning. As a result, the emission reductions saturate when the delay factor is higher than for all alternatives.

Among all alternatives, the TFI has the biggest emission reduction. However, TFI is the ideal case that requires the engine to use the computationally expensive complicated triple injection at all engine speed ranges, which is infeasible since the engine control software task may miss its deadline. The second best is MFI with the heterogeneous speed profile. We find that it outperforms MFI-NS, which coincides with our observations in Sec. VI. MFI also outperforms SFI because SFI only considers single fuel injection and has less design space to optimize. Meanwhile, as the delay factor increases to 2:0, the cost reduction of MFI (46%) becomes close to the cost reduction of the ideal case TFI (50%). This is because a relaxed deadline allows MFI to optimize the speed profile to use the cost-effective triple injection control strategy at most of the road segments, achieving a close-to-ideal performance.

Another interesting observation comes from the comparison between MFI-NS and SFI. Those two methods consider two separate ways to reduce the emission. MFI-NS has cost-effective triple-injection but only drives at a constant speed over one road segment, while SFI considers heterogeneous speed profile but does not have the triple injection strategy. When the deadline is tight (e.g., $\tau = 1:1$), SFI is slightly worse than MFI-NS because there is little design space for heterogeneous speed profile so the benefit of multiple control strategies is larger. When the deadline gets relaxed (e.g., from 1:2 to 2:0), the emission reduction of SFI outperforms the emission reduction of MFI-NS. This is because, with a relaxed deadline, the heterogeneous speed profile has larger design space and its benefit outperforms the benefit of multiple control strategies. It thus also justifies the necessity of heterogeneous speed profiles in environmentally related truck operations.

road segment. It is thus necessary to consider heterogeneous speed profiles in environment-friendly truck operations. An interesting future direction is to study the multiple objectives by jointly considering fuel consumption and emissions. It would also be interesting to jointly optimize the gear position shifting strategy, path planning and speed planning.

Fig. 8: Relative emission reduction of MFI as compared to FAST with different switching speeds. The delay factor is 1.2.

E. Impact of Switching Speed

In this subsection, we study the impact of switching speed for MFI. We set the delay factor to 1:2 for all 1;000 origin-destination pairs and vary the switching speed from 1;000 rpm to 1;200 rpm. Note that a lower switching speed leads to a smaller feasible rpm range for the cost-effective triple injection. The results are presented in Fig. 8. We find that as switching speed increases, the relative emission reduction of MFI decreases because higher switching speeds allow larger engine speed ranges and thus larger vehicle speed ranges for the cost-effective triple injection strategy. Meanwhile, for all switching speeds from 800 to 1;200, the boxes in Fig. 8 have small lengths. This means that the variance of relative emission reduction of MFI is small and MFI has uniformly good performance for all 1;000 origin-destination pairs.

VII. CONCLUSION

In this paper, we consider a scenario where a heavy-duty truck hauls freight across a national highway network subject to a hard deadline. We ride on the recent advancement in engine control that adaptively selects the fuel injection strategy to effectively reduce the emission. We show that the problem is NP-hard, and the adaptive fuel injection strategy imposes a unique challenge compared to existing studies due to the non-convexity of the emission rate function. We reformulate the speed planning problem as a convex problem and leverage the problem structure to compute the optimal solution by checking a small number of candidates. We then propose an efficient heuristic for the overall problem with both path planning and speed planning, and derive an upper bound of the performance gap for the heuristic.

We evaluate the performance of our approach using real-world traces over the US national highway system. The results show that our approach reduces up to 46% emission as compared to the fastest path, which is commonly adopted in practice. We also study the impact of heterogeneous speed profile on one road segment and find that it reduces up to 32% as compared to its counterpart that uses constant speed on one

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VIII. APPENDIX

A. An example of heterogenous speed planning

Example 1. Let us consider the following emission rate function

$$f(r) = \begin{cases} (r - 30)^2 = 100 + 1 & \text{if } 30 \leq r \leq 50; \\ (r - 50)^2 = 100 + 10 & \text{if } 50 < r \leq 60 \end{cases}$$

We further set the length of the edge to 10, and the total time for traversing this edge is 2.

(i) By following a constant speed of $v = 55$, the total emission is

$$2 \cdot f(55) = 20:5$$

(ii) In comparison, consider another solution where we first drive at a speed of 40 for time 0.5, and then drive at a speed of 60 for time 1.5. This solution is feasible since it traverses the edge (with a length of 10) by a total travel time of 2. The incurred emission is

$$0.5 \cdot f(40) + 1.5 \cdot f(60) = 17:5$$

Therefore, driving at a constant speed of 55 incurs a larger emission than the solution (ii); the solution (i) is not optimal.

the following inequality.

$$\sum_{i=1}^n f_i(r_i) \geq \sum_{i=1}^n \tilde{f}_i(r_i) \tag{24}$$

$$= y_1 + y_2 + y_3 \geq \tilde{y}_1 + \tilde{y}_2 \tag{25}$$

$$= \frac{y_3 - y_1}{r_{i_2} - r_{i_1}} (r_{i_2} - r_{i_1}) + (y_2 - y_1) \tag{26}$$

$$= (y_3 - y_1) \frac{y_3 - y_1}{r_{i_3} - r_{i_1}} + \frac{y_2 - y_1}{r_{i_2} - r_{i_1}} \tag{27}$$

$$\geq 0 \tag{28}$$

Here, equality (26) is derived by substituting the solution of equations (23). The last inequality is true because all of the product components in (27) is positive. The above inequality means that the new solution decrease the number of non-zero entries with no larger objective. Therefore, if there is an optimal solution with more than two nonzero entries, we can always follow the above procedure to construct another optimal solution with at most two non-zero entries. \square

Fig. 9: An illustration of Example 1

B. Proof of Lemma 3

Proof. Suppose there is a solution with at least three non-zero entries and its corresponding speed vector v . We will show that we can decrease the non-zero entries by one without loss of optimality. Pick any three of nonzero entries and denote them by $i_1; i_2; i_3 > 0$ with corresponding function values $y_1 = f_{i_1}(r_{i_1}), y_2 = f_{i_2}(r_{i_2}), y_3 = f_{i_3}(r_{i_3})$. Let

$$\begin{aligned} \lambda &= i_1 + i_2 + i_3 \\ \mu &= \frac{i_1 r_{i_1} + i_2 r_{i_2} + i_3 r_{i_3}}{\lambda} \end{aligned}$$

Without loss of generality, we assume the following conditions

$$r_{i_1} < \mu < r_{i_2}; \mu < r_{i_3} \tag{21}$$

$$\frac{y_2 - y_1}{r_{i_2} - r_{i_1}} \geq \frac{y_3 - y_1}{r_{i_3} - r_{i_1}} \tag{22}$$

Note that similar argument holds for the case when r_{i_2} is less than μ or when the slope order in (22) is swapped. Let $\tilde{y}_1; \tilde{y}_2$ to the following equations:

$$\begin{aligned} \lambda &= \tilde{y}_1 + \tilde{y}_2 \\ \mu &= \tilde{y}_1 r_{i_1} + \tilde{y}_2 r_{i_2} \end{aligned} \tag{23}$$

Then we can construct a new solution

$$\tilde{y}_j = \begin{cases} \tilde{y}_1; & \text{if } j = i_1 \\ \tilde{y}_2; & \text{if } j = i_2 \\ 0; & \text{if } j = i_3 \\ y_j; & \text{otherwise} \end{cases}$$

and the same speed profile. By construction, \tilde{y} and μ satisfy the constraints of problem (9). Meanwhile, we have

C. Proof of Lemma 4

Our proof relies on the following lemma, which is also known as chordal slope lemma.

Lemma 8 (Chordal Slope Lemma [57]) Suppose $g(r)$ is a function of one real variable. Consider the slope $S(r_1; r_2) = \frac{g(r_2) - g(r_1)}{r_2 - r_1}$ of the line connecting two points $(r_1; g(r_1))$ and $(r_2; g(r_2))$ on the curve. If g is convex, then $S(r_1; r_2)$ is monotonically non-decreasing in r , for every $x \in [r_1, r_2]$.

Proof of Lemma 4. The intercept point y can be written as

$$\begin{aligned} y &= \frac{y_2 - g(r_1)}{r_2 - r_1} (r - r_1) + g(r_1) \\ &= y_2 + \frac{g(r_2) - g(r_1)}{r_2 - r_1} (r - r_2) + \frac{(r_2 - r)(y_2 - g(r_2))}{r_1 - r_2} \end{aligned}$$

By Lemma 8 and that $(r - r_2)$ is non-positive, the second summand on the right hand side is monotonically non-increasing with r_1 . The third summand is also monotonically non-increasing with r_1 , since both $(r_2 - r)$ and $(y_2 - g(r_2))$ are non-negative. Hence, y is also monotonically non-increasing with r_1 . \square

D. Proof of Lemma 5

We consider the slope of the line connecting $(r_1; y_1)$ and $(r_2; g(r_2))$, which is the major component of

$$T(r_2) = \frac{g(r_2) - y_1}{r_2 - r_1}$$

In the following, we will first prove the following two useful Lemmas before we prove Lemma 5.

Lemma 9. The following two statements are true.

- If $r_1 < r < p$, then $T(r) > T(p)$;
- If $r > p > r_1$, then $T(r) < T(p)$.

Proof. We first consider a help function

$$h(r) = g(r) + \alpha(r)(r_1 - r)$$

where $\alpha(r)$ is the subgradient of function g at point r and both left derivative and right derivative lie in $\alpha(r)$. For any a, b such that $r_1 \leq a \leq b$, we have

$$\begin{aligned} h(b) - h(a) &= g(b) - g(a) + \alpha(b)(r_1 - b) - \alpha(a)(r_1 - a) \\ &= \alpha(b)(b - a) + \alpha(b)(r_1 - b) - \alpha(a)(r_1 - a) \\ &= \alpha(b) - \alpha(a) \geq 0 \end{aligned} \quad (29)$$

The inequality directly follows from the convexity of function g . Therefore, function $h(x)$ is non-increasing when $x \leq r_1$. When $r_1 \leq r \leq p$, we have

$$\begin{aligned} &\alpha(g(r)) - T(r) \\ &= \frac{y_1 - g(r) + \alpha(g(r))(r_1 - r)}{r - r_1} \\ &= \frac{y_1 - g(r) + \alpha(g(p))(r_1 - p)}{r - r_1} \\ &= \frac{r_1 - p}{r - r_1} \frac{y_1 - g(p)}{r_1 - p} - \alpha(g(p)) \\ &= 0 \end{aligned} \quad (30)$$

Here, inequality (31) follows from (29) and inequality (32) follows from the definition of the generalized right point of tangency. Similarly, when $r \geq p \geq r_1$, we have

$$\begin{aligned} &\alpha(g(r)) - T(r) \\ &= \frac{y_1 - g(r) + \alpha(g(r))(r_1 - r)}{r - r_1} \\ &= \frac{y_1 - g(r) + \alpha(g(p))(r_1 - p)}{r - r_1} \\ &= \frac{r_1 - p}{r - r_1} \frac{y_1 - g(p)}{r_1 - p} - \alpha(g(p)) \\ &= 0 \end{aligned}$$

Then we complete our proof for Lemma 9. \square

The following Lemma characterizes the monotonicity of $T(r)$.

Lemma 10. $T(r)$ is first monotonically non-increasing with r before p , then monotonically non-decreasing with r after p .

Proof. When $a \leq b \leq p$, we have

$$\begin{aligned} T(a) &= \frac{g(a) - y_1}{a - r_1} \\ &\stackrel{(1)}{=} \frac{g(b) + \alpha(g(b))(a - b) - y_1}{a - r_1} \\ &\stackrel{(2)}{=} \frac{g(b) + \frac{g(b) - y_1}{b - r_1}(a - b) - y_1}{a - r_1} \\ &= \frac{g(b) - y_1}{b - r_1} = T(b) \end{aligned}$$

Here inequality (1) is because g is convex hence $g(a) \geq g(b) + \alpha(g(b))(a - b)$, and inequality (2) is because α is not larger than the generalized right point of tangency.

Similarly, when $p \leq a \leq b$, we have

$$\begin{aligned} T(b) &= \frac{g(b) - y_1}{b - r_1} \\ &\stackrel{(3)}{=} \frac{g(a) + \alpha(g(a))(b - a) - y_1}{b - r_1} \\ &\stackrel{(4)}{=} \frac{g(a) + \frac{g(a) - y_1}{a - r_1}(b - a) - y_1}{b - r_1} \\ &= \frac{g(a) - y_1}{a - r_1} = T(a) \end{aligned}$$

Then we complete our proof for Lemma 10. \square

Now we proceed to prove Lemma 5.

Proof of Lemma 5. Recall that we can write y as

$$y = \frac{g(r_2) - y_1}{r_2 - r_1}(r - r_1) + y_1$$

By Lemma 10 and that $(r - r_1)$ is non-negative, we have the monotonicity of y follows the monotonicity of $T(r_2)$.

That is, y is monotonically non-increasing before p , then monotonically non-decreasing after p .

If $r \leq p$, since y is monotonically non-increasing with $r_2 \geq r$, thus we have

$$\begin{aligned} y &= y_1 + \frac{g(r_2) - y_1}{r_2 - r_1}(r - r_1) \\ y_1 + \frac{g(r) - y_1}{r - r_1}(r - r_1) &= g(r) \end{aligned}$$

If $r \geq p$, since y is monotonically non-decreasing with $r_2 \geq r$, thus we have

$$\begin{aligned} y &= y_1 + \frac{g(r_2) - y_1}{r_2 - r_1}(r - r_1) \\ y_1 + \frac{g(r) - y_1}{r - r_1}(r - r_1) &= g(r) \end{aligned}$$

\square

\square E. Proof of Proposition 1

Proof. The proof idea follows from [58, Proposition 2.1]. We first rewrite problem in the following compact form

$$c^*(z) = \min_{z \in Z(z)} F(z)$$

where $z = [d_1; \dots; d_n; t_1; \dots; t_n]$ is the stacked variable, and

$$\begin{aligned} Z(z) = \{z : & \sum_{i \in [n]} t_i = \sum_{i \in [n]} d_i = D \\ & s_{i-1} t_i \leq d_i \leq s_i t_i; \forall i \in [n] \} \end{aligned} \quad (34)$$

$$F(z) = \sum_{i \in [n]} t_i f_i \frac{d_i}{t_i} \quad (35)$$

are the constraint set and the objective. By Lemma 2, we know that $F(z)$ is convex in z . Note that for every $z \in [t_l; t_u]$,

