Optimizing Demand Response in Distribution Network with Grid Operational Constraints

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ABSTRACT

Despite the extensive studies on end-user participation in distribution networks, incorporating grid operational constraints and the incentive/dynamic pricing in demand response (DR) is still a challenging and open problem. To fill this gap, we propose a novel three-stage game framework to enable the DR among the utility company, distribution system operator (DSO), and prosumers. In Stage I, utility determines the incentive price to DSO for social welfare maximization. In Stage II, DSO decides the dynamic prices to prosumers and respects grid operational constraints. In Stage III, each prosumer adjusts the local generation and demand on its behalf. We show that the DR game admits an equilibrium that maximizes social welfare and DSO/prosumers' benefits while satisfying operational constraints. We prove the uniqueness of the optimal power supply of utility and the demand-generation adjustments and derive the explicit form of optimal incentive/dynamic price-setting at equilibrium. We further develop a robustness-enhanced design against DSO/prosumers' fault information and explore the impact of renewable/uncontrollable load uncertainty. Meanwhile, we develop an efficient distributed algorithm to help DR participants cooperatively reach equilibrium. Simulations show that the proposed scheme improves social welfare by 20.1% and DSO/prosumers' benefit by 32.5% on IEEE 30/118-bus systems while respecting all grid operational constraints.

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CCS CONCEPTS

• **Hardware** → **Smart grid**; **Power Network**; • **Mathematics of computing** → **Network optimization**; • **Applied computing** → **Multi-criterion optimization and decision-making**; • **Theory of computation** → *Design and analysis of algorithms.*

KEYWORDS

Demand response, System operational constraints, Pricing design, Multi-stage optimization and game

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1 INTRODUCTION

The fast deployment of cloud computing and data centers has contributed significantly to the internet of things applications in daily life, e.g., transportation and power systems [53]. Though the penetration of renewables helps to decarbonize the increasing energy consumption of large-scale cloud service providers and other consumers, the uncertain demand/supply imposes unprecedented challenges in the efficient operation of power systems [23, 35, 44, 52]. With the great advances in communication and control in smart grid technologies, demand response (DR) has been proved as a promising solution by actively involving users in the demand side management for peaking load shaving [26] and elastic demands and fluctuating generations matching [17]. DR allows the utility company to manage end-users' energy consumption, either directly (via remote load control) or indirectly (via pricing schemes), to improve power system efficiency; see [11] for a comprehensive survey.

In this work, we consider the DR between the utility, the nonprofit organization named distribution systems operator (DSO) who manages the distribution network, and the prosumers in the same

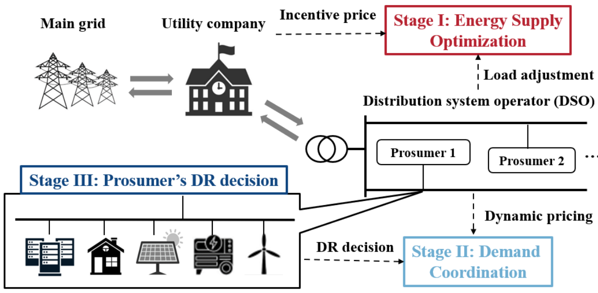


Figure 1: Structure of demand response scheme.

network. Prosumers are precipitated from consumers with the capability of local generation, e.g., micro-grids and data centers. Viewing the prominent role of game theory in smart grids [34], we model their interactions as a three-stage game, which is in a hierarchical structure such that utility offers incentive prices and in response, DSO provides the dynamic prices to prosumers to guide them in adjusting the demand and local generations while respecting the network constraints; see Fig. 1 for an illustration. We remark that the game-theoretical model is widely adopted in DR schemes. e.g., the two-stage Stackelberg model for energy consumers [20, 21, 24], energy storage systems [17], and data centers [39]. Such paradigm is known as the *Price-based approach* as discussed in Sec. 2.

Although extensive studies have been conducted on developing different DR schemes, there are still several challenges in designing an optimal and feasible DR mechanism.

- First, it is crucial to incorporate the power network constraints into the design, e.g., the branch flow and demand-generation limits and the power balance equations.

A major concern of existing approaches lies in ignoring the underlying power network operational constraints but only considering the aggregate load-supply balance [20, 27, 32], which could lead the obtained DR solutions infeasible and violate the system constraints severely, e.g., over 94% maximum violation of branch flow constraints in simulation in Sec. 7.

- Second, the DR (game) interactions between utility, DSO, and prosumers have not been fully investigated. For example, existing work mainly studies the DR either between DSO and prosumers or requires the utility directly control the energy consumption/generation in distribution network [21, 34, 51].
- Third, it is non-trivial to analyze the DR game equilibrium and obtain the optimal incentive/dynamic price-setting schemes, i.e., equilibrium and the corresponding pricing schemes are expected to be robust and maximize social welfare so that the system regulator can advocate DR with legitimate support.

Incorporating the system operational constraints and the three-stage game model render the analysis of the DR scheme challenging due to the lack of closed-form optimal decisions. However, the DR equilibrium properties and the corresponding optimal price-setting have not been fully explored in existing work, e.g., [21, 30, 45, 51].

In this work, we carry out a comprehensive study to tackle these challenges and make the following contributions.

▷ After modeling power network and DSO/prosumer's behaviors in Sec. 3, we develop a price-based DR framework between utility, DSO, and prosumers as a three-stage game considering system operational constraints in Sec. 4. We show the existence and efficiency of the equilibrium, i.e., social welfare at equilibrium is the same as the maximum one under coordinated setting. The uniqueness of optimal power supply and demand/local generation adjustments is also proved. We show that instead of solving the non-convex multi-level DR game problem, one can solve the convex social welfare maximization to derive the identical equilibrium DR solution.

▷ In Sec. 5.2, based on the understanding of reacting to electricity market marginal price and network locational marginal price, we present the closed-form optimal incentive/dynamic price-setting schemes to guarantee the efficient equilibrium. To our best knowledge, it is the first comprehensive pricing result considering power network operational constraints in the three-stage DR game model. Furthermore, in Sec. 5.3, we present a robustness-enhanced design to the scheme. That is, any fault information from DSO/prosumers will not change the optimal incentive/dynamic prices, and DR still maximizes social welfare. We further analyze the impact of renewable/uncontrollable load uncertainty. It is shown that the fluctuation of renewables/uncontrollable loads will poison network efficiency and DSO and prosumers' benefits.

▷ To further meet the privacy preservation requirement in scheme implementation, in Sec. 6, we explore an efficient and computationally tractable distributed algorithm to determine the equilibrium. Based on the projected gradient descent method, we decouple the optimization variables of utility and DSO/prosumers and allow them to solve the concerned program concurrently, which is guaranteed to converge to the equilibrium set point.

Simulation results in Sec. 7 show that the proposed DR scheme increases social welfare by 20.1% and DSO/prosumers' benefit by 32.5% on IEEE Case30/118-bus systems. In addition, DR solutions respect all constraints, which surpasses the scheme ignoring network constraints causing >94.0% maximum violations. For fluency, all proofs are presented in the appendix.

2 RELATED WORK

Generally, there are two *orthogonal* categories of residential DR programs differentiated by their coordination approaches.

(i) *Direct load control (DLC):* The DLC approaches implemented by utility are usually contract-based, in which consumers allow the utility to adjust the energy usage of the household appliances remotely [34]. The rewards to consumers participating in DLC programs for load shedding can be pre-set [40] or event-orientated [2]. An online task scheduling algorithm to control data centers for DR is proposed in [37]. In [18], demand response and battery storage are jointly optimized from the utility standpoint. Utilities may also design new electricity products to directly elicit load flexibility [25]. Meanwhile, system operational constraints can be incorporated [51] into DLC-based DR schemes.

(ii) *Price-based approaches:* In contrast, *price-based approaches* allow utility to indirectly control consumers' loads by providing incentive prices for desired load shifting goals and thus achieve better privacy preservation. These approaches include various pricing schemes, e.g., real-time, time-of-use, and critical peak pricing [3] based on different criteria. Several game-theoretical frameworks based on the Stackelberg game [21, 45], mean-field game [10], and others [13] are developed. Works [21, 30, 45, 51] examine the power network constraints in DR, though most existing results ignore such operational limits. In [21], the DR between utility and consumers is explored but requires utility to access full power network details and ignores local generations and dynamic electricity market prices. The incentive/dynamic price-setting schemes among utility, DSO, and prosumers are also not fully investigated. In [6], a gradient-based approach is proposed to model the energy consumption response to electricity prices based on historical prices and response data. [43] proposed an online price-based control method for DR, which can achieve a provable competitive ratio for peak demand under future load uncertainty. Study [30] presents the DR incorporating the distribution locational marginal price and PV uncertainty. In [7, 8, 46, 53], the authors show that data centers can be applied for DR via auction mechanism and pricing approaches.

Besides these two categories, there is also a line of works focusing on the energy trading/sharing between prosumers in the same region [9, 19, 42] to lower their costs. For example, a direct energy trading framework among prosumers based on the Nash bargaining is provided in [19]. These works indicate the potential of energy trading/sharing among prosumers in DR but ignore the game-theoretical interactions with DSO and utility.

However, these works have not fully addressed the challenges discussed in Sec. 1, i.e., obtaining the DR equilibrium and the optimal pricing schemes considering power network constraints. In this paper, we study the DR between utility, DSO, and prosumers based on the three-stage model as the *Price-based approach* and tackle these challenges. Our work differs from existing literature in that we include power network constraints into the design to guarantee the scheme's feasibility and practicability and fully investigate the equilibrium existence and efficiency under the novel three-stage game-theoretical model. Furthermore, we derive the closed-form optimal incentive/dynamic price-setting schemes, a robustness-enhanced design under the fault-ridden setting, and an efficient distributed algorithm while respecting all system constraints. The impact of renewable//uncontrollable load uncertainty is also investigated.

3 SYSTEM MODEL

In this section, we present the model of the utility company and DSO with prosumers deployed in a distribution network. Key symbols are summarized in Table 1.

Table 1: Key notations

Notation	Definition
$\mathcal{B}/\mathcal{N}$	Set of buses/prosumers, $B \triangleq \lvert\mathcal{B}\rvert$, $N \triangleq \lvert\mathcal{N}\rvert$
$\mathcal{M}$	Set of generators of utility, $M \triangleq \lvert\mathcal{M}\rvert$
P_{buy}	Per unit purchasing price from utility.
P_{sell}	Per unit selling-back price to utility (feed-in tariff)
$z^{\text{D-U}}/z^{\text{U-G}}$	Power from utility to DSO/grid to utility
$z^{\text{D-U}}_{\max}/z^{\text{D-U}}_{\min}$	Upper/lower bound of $z^{\text{D-U}}$
a_i^D, b_i^D	Generation cost function coefficients of prosumer i
a_i^U, b_i^U	Generation cost function coefficients of utility
x_i^D/x_j^U	Local energy generation of prosumer i/utility
D_i/U_i	Controllable/Uncontrollable load of prosumer i
r_i/e_i	Prosumer $i's$ renewable generation/$e_i = r_i - U_i$
$\bar{D}_i/\underline{D}_i$	Upper/lower bound of D_i
p_i/D_i^{Ad}	Generation/demand adjustment of prosumer i
$\mathcal{E}, \mathbf{L}$	Set of branches and the transmission limits, $E \triangleq \lvert\mathcal{E}\rvert$
$\bar{x}_i^D/\underline{x}_i^D$	Upper/lower bound of x_i^D

Note: we use $|\cdot|$ to denote the size of a set.

3.1 Distribution Network Model

We consider the distribution power network as illustrated in Fig. 1 in which prosumers are interconnected. The DSO is located in the slack bus through which the energy can be traded with the utility. The power network constraints are

$$z^{\text{D-U}} + \sum_{i\in\mathcal{N}} x_i^D + \sum_{i\in\mathcal{N}} r_i = \sum_{i\in\mathcal{N}} D_i + \sum_{i\in\mathcal{N}} U_i, \tag{1}$$

$$-\mathbf{L} \le \mathbf{PTDF} \cdot (\mathbf{x}^D + \mathbf{r} - \mathbf{D} - \mathbf{U}) \le \mathbf{L}. \tag{2}$$

(1) is the power balance equation. (2) are the branch flow limits. Here $\mathbf{x}^D \in \mathcal{R}^{B-1}, \mathbf{D} \in \mathcal{R}^{B-1}$ and $x_i^D = D_i = 0$ if $i \notin \mathcal{N}$. The Power Transfer Distribution Factors [48] matrix $\mathbf{PTDF} \in \mathcal{R}^{E\times(B-1)}$ depends on the power network topology. We remark that the above direct current power flow formulation is widely adopted in literature [38, 49]. For simplicity, we use $e_i = r_i - U_i$ to denote the net uncontrollable power injection at node i.

3.2 DSO and Prosumer's Model

Prosumers interconnected in the distribution power network are modeled to be managed by DSO.[1] We assume that each prosumer $i \in \mathcal{N}$ is equipped with the distributed generator.[2] The goal of DSO, who is the central operator to coordinate the energy trading with utility, distributed power generation, and energy consumption of all prosumers, is to maximize total social welfare while satisfying all power network operational constraints.

3.2.1 Energy Trading with Utility. Note that $z^{\text{D-U}}$ is the energy traded with the utility. P_{buy} and P_{sell} (\$/MWh) denote the purchasing price and the selling-back price with utility,[3] Here P_{buy} and P_{sell} are commonly contractual set and are modelled to be invariant w.r.t. the traded quantity. Due to the physical or contractual limits, we have

$$z^{\text{D-U}}_{\min} \le z^{\text{D-U}} \le z^{\text{D-U}}_{\max}. \tag{3}$$

Therefore, DSO's cost to utility is given as:

$$C_u(z^{\text{D-U}}) = P_{\text{buy}} \cdot \max(z^{\text{D-U}}, 0) + P_{\text{sell}} \cdot \min(z^{\text{D-U}}, 0). \tag{4}$$

In practice, $P_{\text{buy}} \ge P_{\text{sell}}$ to make the feed-in tariff feasible [22].

[1] Here we consider each prosumer is located in a single bus, which can be extended such that a prosumer corresponds to multiple buses.

[2] In the short-term time scale, renewable generations and uncontrollable loads can be predicted reasonably well and we assume $r_i, i \in \mathcal{N}$ have zero marginal cost [41]. For simplicity, we first consider fixed renewable/uncontrollable injections. The impact of renewable/uncontrollable load uncertainty is further studied in Sec. 5.4.

[3] DSO can sell its surplus energy when total local generation exceeds total demand, especially considering renewables. P_{sell} is known as the feed-in tariff.

3.2.2 Local Generation Cost and Benefit Function. Prosumers are assumed to be equipped with local generators. Note that the local generation x_i^D of prosumer i is bounded:

$$\underline{x}_i^D \le x_i^D \le \bar{x}_i^D, \quad i \in \mathcal{N}. \tag{5}$$

The generation cost function is usually modeled to be strictly convex subdifferentiable, e.g., the quadratic functions in [31]:

$$C_{g,i}^D(x_i^D) = a_i^D \cdot {x_i^D}^2 + b_i^D \cdot x_i^D, \quad i \in \mathcal{N}, \tag{6}$$

where a_i^D and b_i^D are the positive generation coefficients. We further model the benefit of prosumers to consume energy. Recall that D_i is its demand, which is bounded:

$$0 \le \underline{D}_i \le D_i \le \bar{D}_i, \quad i \in \mathcal{N}. \tag{7}$$

Prosumer $i's$ benefit function B_i is modeled to be strictly concave and differentiable. A common choice of B_i is logarithmic function with preference coefficients $k_i > 0, \delta_i \ge 1$:

$$B_i(D_i) = k_i \ln(\delta_i + D_i) + k_i^u \ln(\delta_i^u + U_i), \quad i \in \mathcal{N}. \tag{8}$$

The first and second term in (8) denotes the benefit of controllable and uncontrollable load respectively. Note that if δ_i(resp δ_i^u) < 1 or k(resp k_i^u) < 0, $B_i(D_i)$ can be negative, which is not realistic in practice. The logarithmic function (8) is widely applied to model prosumer's economic behavior [4, 14] and has been validated in different DR schemes [14, 15, 24]. We remark that our analysis holds for general strictly concave subdifferentiable $B_i(D_i)$. For prosumers without local generation (resp demand), one can simply set $(\underline{x}_i^D, \bar{x}_i^D, a_i^D, b_i^D) = \mathbf{0}$ (resp $(\underline{D}_i, \bar{D}_i, k_i) = \mathbf{0}, \delta_i = 1$) to make the formulation consistent.

3.2.3 Total Benefit of DSO and Local Optimization. The total benefit of DSO is the aggregate net benefit of all prosumers:

$$B_{\text{total}}^D = \sum_{i \in \mathcal{N}} (B_i(D_i) - C_{g,i}^D(x_i^D)) - C_u(z^{\text{D-U}}). \tag{9}$$

Before involving in DR, DSO focuses on the following local benefit maximization considering all power network constraints:

P0: DSO's Optimization without Demand Response

$$\begin{aligned} \max \quad & B_{\text{total}}^D(z^{\text{D-U}}, \mathbf{x}^D, \mathbf{D}) \\ \text{s.t.} \quad & (1) - (3), (5), (7), \\ \text{var.} \quad & z^{\text{D-U}}, \mathbf{x}^D, \mathbf{D}. \end{aligned}$$

Let $(z_*^{\text{D-U}}, \mathbf{x}^{D*}, \mathbf{D}^*)$ denote the optimum of **P0**. The following lemma shows that the optimal supply from utility and local generation and demand can be uniquely determined before DR in **P0**.

Lemma 1. ***P0** is convex and admits a unique optimal solution.*

3.2.4 Dynamic Pricing to Each Prosumer. In practice, DSO may not be able to directly control the energy consumption and local generation of prosumers. In this work, we model the *dynamic pricing* from DSO to coordinate prosumers' behaviors. Here we consider two factors: the unit price of electricity consumption P_i^{dp}, and the distribution network access fee of prosumer i denoted by k_i. We remark that the implemented price signal (P_i^{dp}, k_i) to each prosumer represents the actual price of electricity consumption or is just a control signal to coordinate users' decisions. Therefore, each prosumer i chooses the optimal power profile (D_i, x_i^D) to maximize its individual net benefit, i.e., benefit minus payment:

P0-sub: Prosumer's optimal energy decision before DR

$$\begin{aligned} \max_{D_i, x_i^D} \quad & B_{pro,i} = B_i(D_i) - C_{g,i}^D(x_i^D) - P_i^{dp} \cdot (D_i - x_i^D - e_i) - k_i \\ \text{s.t.} \quad & (5), (7). \end{aligned}$$

Here $P_i^{dp} \cdot (D_i - x_i^D - e_i) + k_i$ are the payment to DSO, which can be positive/negative, representing the monetary paid to/received from DSO for energy consumption and generation. Here the value of k_i is not related to the choice of (D_i, x_i^D). In practice, the dynamic pricing scheme is expected to have the following properties:

- Social welfare maximization: the induced decisions of prosumers $(\mathbf{x}^{\mathbf{D}}, \mathbf{D})$ should maximize the total benefit B_{total}^D and respect the distribution network constraints;
- DSO's revenue maximization: given prices (P_i^{dp}, k_i), the individual decisions from prosumers should maximize DSO'S total revenue, i.e., same as the coordinated solution of

$$\begin{aligned} \max_{z^{\text{D-U}}, \mathbf{x}^D, \mathbf{D}} \quad & \sum_{i \in \mathcal{N}} (P_i^{dp} \cdot (D_i - x_i^D - e_i) + k_i) - C_u(z^{\text{D-U}}) \\ \text{s.t.} \quad & (1) - (3), (5), (7). \end{aligned}$$

- Budget balance: the total payments received by DSO from prosumers should be no more than the payment to utility, i.e., $\sum_{i \in \mathcal{N}} (P_i^{dp} \cdot (D_i - x_i^D - e_i) + k_i) \le C_u(z^{\text{D-U}})$.

The above three conditions represent the validity of the dynamic pricing scheme in implementation. We have the following result:

Lemma 2. *The dynamic pricing scheme satisfies the above three conditions if for each $i \in \mathcal{N}$, (P_i^{dp}, k_i) is set as*

$$P_i^{dp} = -\eta_0 - \sum_{l=1}^{E} (\bar{\epsilon}_{l,0} a^{(l,i)} - \underline{\epsilon}_{l,0} a^{(l,i)}), \tag{10}$$

$$k_i = \frac{P_I \sum_{j \neq i} D_{j,0}^{net*} - \sum_{j \neq i} P_j^{dp} D_{j,0}^{net*}}{N-1}. \tag{11}$$

*Here $P_I = P_{buy}$ if $z_*^{D\text{-}U} \ge 0$ and $P_I = P_{sell}$ if $z_*^{D\text{-}U} < 0$. The net demand $D_{i,0}^{net*} = D_i^* - x_i^{D*} - e_i$. $(x_i^{D*}, D_i^*, z_*^{D\text{-}U} = \sum_{i \in \mathcal{N}} D_{i,0}^{net*})$ and $(\eta_0, \bar{\epsilon}_{l,0}, \underline{\epsilon}_{l,0})$ are the optimal solutions and the corresponding Lagrangian dual variables of the KKT conditions for **P0** as discussed in Appendix B.1 respectively.*

We remark that the unit price P_i^{dp} to prosumers can be interpreted as the *Locational Marginal Price* at each node of the power system giving network constraints and objective, which represents the network operation/congestion conditions [38]. $k_i, i \in \mathcal{N}$ are designed to maintain the budget balance of DSO. In Sec. 5.2, we further show the optimal dynamic pricing *considering* the incentive pricing from utility in the DR program. After determining the dynamic pricing to each prosumer, DSO then passes the optimum $z_*^{\text{D-U}}$, i.e., the aggregated net load, to utility for further DR implementation.

4 OPTIMAL THREE-STAGE DR GAME

In this section, we propose a novel DR scheme based on the three-stage game-theoretical model. After receiving DSO's pre-DR solution of **P0**, in Stage I, utility determines the optimal power supply

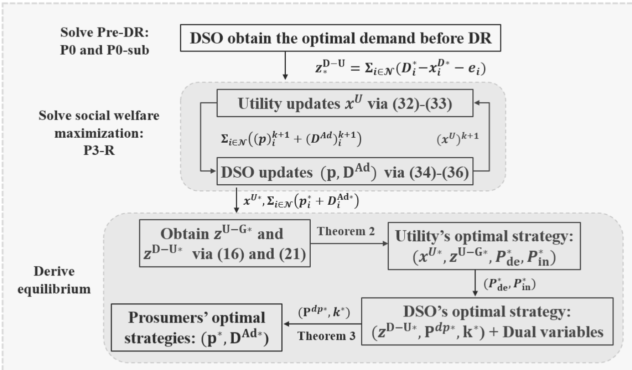


Figure 2: Structure of the proposed demand response scheme

and provides the incentive price to induce DSO/prosumers' load adjustments. In Stage II, given the incentive price, DSO provides the dynamic prices to prosumers to coordinate their DR decisions while respecting network constraints. Finally, in Stage III, given DSO's dynamic pricing, each prosumer chooses its individual optimal decision; see Fig. 2 for illustration. Utility is assumed to be regulated such that it aims to maximize social welfare instead of its profit by selling power when designing strategy [20]. Next, we use backward induction to derive their optimal strategies and the DR equilibrium.

4.1 Stage III: Prosumers' Optimal DR Decision

In DR, after receiving DSO's dynamic pricing, each prosumer determines the optimal local generation and demand adjustment. The generation adjustment p_i is bounded:

$$\underline{p}_i \le p_i \le \bar{p}_i, \quad i \in \mathcal{N}, \tag{12}$$

where $\bar{p}_i$ and $\underline{p}_i$ are the upper and lower bound of p_i separately. Therefore, the generation cost after such output adjustment is

$$\tilde{C}_{g,i}^D(p_i) = C_{g,i}^D(x_i^{D*} + p_i), \quad i \in \mathcal{N}. \tag{13}$$

Similarly, the demand adjustment D_i^{Ad} is bounded:[4]

$$\underline{D}_i^{\text{Ad}} \le D_i^{\text{Ad}} \le \bar{D}_i^{\text{Ad}}, \quad i \in \mathcal{N}, \tag{14}$$

and the benefit function is given as:

$$\tilde{B}_i(D_i^{\text{Ad}}) = B_i(D_i^* - D_i^{\text{Ad}}), \quad i \in \mathcal{N}. \tag{15}$$

Positive p_i and D_i^{Ad} means that prosumer i will increase its local generation and decrease its energy usage, resulting the decrease in net demand. Similarly, negative p_i and D_i^{Ad} indicates the increase of net demand. Therefore, prosumer i optimizes:

DR-sub: Prosumer's optimal energy decision after DR

$$\max_{p_i, D_i^{\text{Ad}}} \quad B_{pro,i}^{\text{DR}} = \tilde{B}_i(D_i^{\text{Ad}}) - \tilde{C}_{g,i}^D(p_i) - P_i^{dp} \cdot D_i^{\text{net}} - k_i$$
$$\text{s.t.} \quad (12), (14), D_i^{\text{net}} = D_i^* - D_i^{\text{Ad}} - x_i^{D*} - p_i - e_i.$$

Here (P_i^{dp}, k_i) can be different from the one in **P0-sub** for different objectives of DSO. Lemma 3 shows that a larger unit price P_i^{dp} will incur a larger load and local generation adjustment.

Lemma 3. ***DR-sub*** *is convex and admits the unique optimal* (D_i^{Ad*}, p_i^*)*. In addition, for any* $P_i^{dp,1} \ge P_i^{dp,2}$*, the corresponding optimal* $p_i^*|_{P_i^{dp,1}} \ge p_i^*|_{P_i^{dp,2}}$ *and* $D_i^{Ad*}|_{P_i^{dp,1}} \ge D_i^{Ad*}|_{P_i^{dp,2}}$*.*

4.2 Stage II: DSO's Dynamic Pricing

The DSO serves as the intermediary between utility and prosumers to 1) maintain the distribution network stability as each prosumer's decision does not consider the network constraints, 2) maximize total regional welfare, 3) maximize its own revenue. In particular, the induced prosumers' DR adjustments should be feasible and satisfy (3) and the following constraints:

$$z^{\text{D-U}} + \sum_{i\in\mathcal{N}} (x_i^{D*} + p_i + D_i^{Ad}) + \sum_{i\in\mathcal{N}} e_i = \sum_{i\in\mathcal{N}} D_i^*, \tag{16}$$

$$-\mathbf{L} \le \mathbf{PTDF} \cdot (\mathbf{x}^{D*} + \mathbf{e} + \mathbf{p} - \mathbf{D}^* + \mathbf{D}^{\text{Ad}}) \le \mathbf{L}. \tag{17}$$

After such adjustments, DSO receives a reward from the utility for decreasing/increasing the aggregated energy usage:

$$R(z^{\text{D-U}}) = P_{\text{de}} \cdot \max(z_*^{\text{D-U}} - z^{\text{D-U}}, 0) + P_{\text{in}} \cdot \max(z^{\text{D-U}} - z_*^{\text{D-U}}, 0). \tag{18}$$

$(P_{\text{de}}, P_{\text{in}})$ are the incentive prices provided by utility, where $P_{\text{de}} > 0$ (resp $P_{\text{in}} > 0$) indicates utility would like the DSO to decrease (resp increase) the energy usage. Here $z_*^{\text{D-U}} - z^{\text{D-U}} = \sum_{i\in\mathcal{N}}(p_i + D_i^{\text{Ad}})$ representing the aggregated load decrease in the network. In practice, utility will only provide one positive incentive price, i.e., $P_{\text{de}}P_{\text{in}} = 0$.[5] Therefore, the total benefit of DSO in the DR is given as:

$$\tilde{B}_{\text{total}}^D = \sum_{i\in\mathcal{N}} (\tilde{B}_i(D_i^{\text{Ad}}) - \tilde{C}_{g,i}^D(p_i)) - C_u(z^{\text{D-U}}) + R(z^{\text{D-U}}),$$

and DSO solves the following optimization problem:

P1: DSO's Dynamic Pricing Setting Optimization

$$\max \quad \tilde{B}_{\text{total}}^D(z^{\text{D-U}}, \mathbf{p}, \mathbf{D}^{\text{Ad}})$$
$$\text{s.t.} \quad (3), (16), (17), (p_i, D_i^{\text{Ad}}) = \arg \mathbf{DR\text{-}sub}|_{(P_i^{dp}, k_i)}, i \in \mathcal{N}$$
$$\text{var.} \quad z^{\text{D-U}}, P_i^{dp}, k_i, i \in \mathcal{N}.$$

In **P1**, DSO determines the optimal dynamic prices (P_i^{dp}, k_i) to prosumers to induce their energy consumption/generation decisions. We remark that the solution of **P1** represents 1) the optimal prosumers' energy profile and 2) DSO's optimal pricing to maximize its benefit, which is the Stackelbergy equilibrium in the DSO-Prosumers game.

4.3 Stage I: Utility's Incentive Price Offering and Energy Supply Optimization

The output of utility's generator x_j^U is bounded:

$$\underline{x}_j^U \le x_j^U \le \bar{x}_j^U, \quad j \in \mathcal{M}. \tag{19}$$

[4] $\bar{p}_i$ and $\underline{p}_i$ can be set as $\bar{x}_i^D - x_i^{D*}$ and $\underline{x}_i^D - x_i^{D*}$. $\bar{D}_i^{\text{Ad}}$ and $\underline{D}_i^{\text{Ad}}$ can be set as $D_i^* - \underline{D}_i$ and $D_i^* - \bar{D}_i$. $p_i = 0$ and $D_i^{\text{Ad}} = 0$ if $i \notin \mathcal{N}$.

[5] Note that $(P_{\text{de}}, P_{\text{in}})$ are the reward-based incentive prices for DR while $(P_{\text{buy}}, P_{\text{sell}})$ are the contract-based flat prices for energy purchasing/selling.

The associated cost function $C_{g,j}^U(x_j^U)$ is also modeled to be strictly convex subdifferentiable as (6), e.g., quadratic:[6]

$$C_{g,j}^U(x_j^U) = a_j^U \cdot {x_j^U}^2 + b_j^U \cdot x_j^U, \quad j \in \mathcal{M}. \tag{20}$$

As discussed before, the utility company is regulated such that it aims to maximize the social welfare defined as

$$B_{\text{social}} = \sum_{i\in\mathcal{N}} (\tilde{B}_i(D_i^{\text{Ad}}) - \tilde{C}_{g,i}^D(p_i)) - f(z^{\text{U-G}}) - \sum_{j\in\mathcal{M}} C_{g,j}^U(x_j^U),$$

where $f(z^{\text{U-G}})$ is the cost function of utility purchasing power from the power market (main grid) to maintain power balance:

$$z^{\text{U-G}} + \sum_{j\in\mathcal{M}} x_j^U + \sum_{i\in\mathcal{N}} (x_i^{D*} + p_i + D_i^{\text{Ad}} + e_i) = \sum_{i\in\mathcal{N}} D_i^*. \tag{21}$$

$f(z^{\text{U-G}})$ can be commonly modeled to be convex and quadratic or piece-wise linear [36, 50]. In this work, we model $f(z^{\text{U-G}})$ to be subdifferentiable convex. Utility then focuses on the following multi-stage game optimization (Stage I):

P2: Utility's Incentive Price Offering and Power Supply

$$\begin{aligned} \max \quad & B_{\text{social}}(z^{\text{U-G}}, \mathbf{x}^U, P_{\text{de}}, P_{\text{in}}, \mathbf{p}, \mathbf{D}^{\text{Ad}}) \\ \text{s.t.} \quad & (19), (21), (\mathbf{p}, \mathbf{D}^{\text{Ad}}) = \arg \mathbf{P1}|_{(P_{\text{de}}, P_{\text{in}})}, \\ & P_{\text{de}} \geq 0, P_{\text{in}} \geq 0, P_{\text{de}}P_{\text{in}} = 0, \\ \text{var.} \quad & z^{\text{U-G}}, \mathbf{x}^U, P_{\text{de}}, P_{\text{in}}. \end{aligned} \tag{22}$$

In **P2**, utility determines the optimal incentive prices $(P_{\text{de}}^*, P_{\text{in}}^*)$ to DSO and its power generation/trading with grid $(z^{\text{U-G}*}, \mathbf{x}^{U*})$ to maximize social welfare using the price-based approach. We remark that the solution of **P2** constitutes the equilibrium of the DR game.

Note that both **P1** and **P2** are non-convex multi-level optimizations that are challenging to solve. In Sec. 5, we show that solving **P1** and **P2** is equivalent to solving a convex social welfare maximization to derive the identical equilibrium optimal power supply and load adjustments $(z^{\text{U-G}*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ and propose an optimal incentive/dynamic price-setting scheme to obtain the equilibrium $(P_{\text{de}}^*, P_{\text{in}}^*)$ and $(P_i^{dp*}, k_i^*, i \in \mathcal{N})$; see Def. 1 for formal definition.

4.4 DR under Coordinated Setting: Overall Energy Consumption Coordination

To further study the social efficiency of the proposed DR scheme, we consider the following two overall energy coordination problems for the centralized optimal decisions of utility and DSO respectively:

4.4.1 *Utility's overall coordination problem.*

P3-U: Social Welfare Maximization Optimization

$$\begin{aligned} \max \quad & B_{\text{social}}(z^{\text{U-G}}, \mathbf{x}^U, \mathbf{p}, \mathbf{D}^{\text{Ad}}) \\ \text{s.t.} \quad & (12), (14), (17), (19), (21), \\ & z_{\min}^{\text{D-U}} \leq z^{\text{U-G}} + \sum_{j\in\mathcal{M}} x_j^U \leq z_{\max}^{\text{D-U}}, \\ \text{var.} \quad & z^{\text{U-G}}, \mathbf{x}^U, \mathbf{p}, \mathbf{D}^{\text{Ad}}. \end{aligned} \tag{23}$$

(23) is the energy trading limit between DSO and utility reformulated from (3) and (21). We have the following lemma.

[6]Our results holds for general convex subdifferentiable $C_{g,j}^U(x_j^U)$. For ease of analysis, we focus on the strictly convex $C_{g,j}^U(x_j^U)$ first.

Lemma 4. ***P3-U** is convex and admits a unique optimal solution.*

Remarks: **P3-U** denotes the optimal decisions of utility under the centralized case that it has full access to the power network. Let $(z^{\text{U-G}*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ denote the optimum of **P3-U**. The optimal solution of **P3-U** may not be identical to that under the game-theoretical setting of **P2**, except for the case that the part optimal solution of **P3-U** $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ happens to be the solution **DR-sub** for each prosumer under some (P_j^{dp}, k_i) and optimal to **P1** and $(z^{\text{U-G}*}, \mathbf{x}^{U*})$ is optimal to **P2** under some $(P_{\text{de}}, P_{\text{in}})$. The difference in the optimal values between **P3-U** and **P2** interprets the loss of efficiency as the price-based mechanism may not guarantee the global optimality of the objective. Nevertheless, in Sec. 5, we show that the equilibrium (solution of **P2**) does not incur efficiency loss.

4.4.2 *DSO's coordination problem.*

P3-D: Distribution Network Benefit Maximization

$$\begin{aligned} \max \quad & \tilde{B}_{\text{total}}^D(z^{\text{D-U}}, \mathbf{p}, \mathbf{D}^{\text{Ad}}) \\ \text{s.t.} \quad & (3), (12), (14), (16), (17), \\ \text{var.} \quad & z^{\text{D-U}}, \mathbf{p}, \mathbf{D}^{\text{Ad}}. \end{aligned}$$

The following lemma shows the property of **P3-D**.

Lemma 5. ***P3-D** admits a unique optimal solution. In addition,*

- *for any $P_{de}^1 \geq P_{de}^2 \geq 0, P_{in} = 0$, the corresponding optimal $\sum_{i\in\mathcal{N}}(p_i^* + D_i^{Ad*})|_{P_{de}^1} \geq \sum_{i\in\mathcal{N}}(p_i^* + D_i^{Ad*})|_{P_{de}^2} \geq 0$;*
- *for any $P_{in}^1 \geq P_{in}^2 \geq 0, P_{de} = 0$, the corresponding optimal $\sum_{i\in\mathcal{N}}(p_i^* + D_i^{Ad*})|_{P_{in}^1} \leq \sum_{i\in\mathcal{N}}(p_i^* + D_i^{Ad*})|_{P_{in}^2} \leq 0$.*

*Furthermore, DSO's maximum benefit after DR, i.e., the optimal objective of **P3-D**, is non-decreasing w.r.t. P_{de} and P_{in}.*

Remarks: **P3-D** represents the coordinated scenario of prosumers as the optimal centralized operation for DSO given $(P_{\text{de}}, P_{\text{in}})$. Note that **P3-D** is non-convex due to the non-concave $\tilde{B}_{\text{total}}^D$ as the piece-wise linear function (18) is convex. Nevertheless, Lemma 5 states that given any incentive price $(P_{\text{de}}, P_{\text{in}})$, the (centralized) unique power supply from utility and generation/demand adjustments can be determined by DSO. It also suggests that a larger incentive price P_{de} (resp P_{in}) encourages DSO to decrease (resp increase) net load more significantly for a larger benefit. As proved in Appendix C.2, solving **P3-D** is equivalent to solving a convexified reformulation of it, with which the uniqueness of solution can also be understood.

5 DR GAME EQUILIBRIUM ANALYSIS

In this section, we show the existence and efficiency of the three-stage DR equilibrium and prove the uniqueness of the optimal power supply and DR adjustments. We provide utility's optimal incentive price-setting considering DSO's response and DSO's dynamic price-setting considering each prosumer's DR decision. We further propose a robustness-enhanced design against DSO/prosumers' fault information and investigate the impact of renewable and uncontrollable load uncertainty.

5.1 Existence and Efficiency of the Equilibrium

We first provide the formal game-theoretical definition of the three-stage game and equilibrium in the following.[7]

- Players: Utility (Stage I), DSO (Stage II), Prosumers (Stage III).[8]
- Strategy: Utility chooses $(z^{\text{U-G}}, \mathbf{x}^U, P_{\text{de}}, P_{\text{in}})$, DSO determines $(z^{\text{D-U}}, P_i^{dp}, k_i, i \in \mathcal{N})$, and prosumers determines $(\mathbf{p}, \mathbf{D}^{\text{Ad}})$.[9]
- Payoff function: Utility maximizes B_{social} in **P2**, DSO maximizes $\tilde{B}^D_{\text{total}}$ in **P1**, and prosumers maximize $B^{\text{DR}}_{pro,i}$ in **DR-sub**.

The corresponding three-stage game equilibrium is defined as:

Definition 1. *Utility's strategy $(z^{U\text{-}G*}, \mathbf{x}^{U*}, P^*_{de}, P^*_{in})$, DSO's strategy $(z^{D\text{-}U*}, P_i^{dp*}, k_i^*, i \in \mathcal{N})$, and prosumers' responses $(\mathbf{p}^*, \mathbf{D}^{Ad*})$ are in the equilibrium if $(\mathbf{p}^*, \mathbf{D}^{Ad*})$ is optimal to each **DR-Sub**, $(z^{D\text{-}U*}, P_i^{dp*}, k_i^*, i \in \mathcal{N})$ is optimal to **P1**, and any other utility's strategy $(z^{U\text{-}G}, \mathbf{x}^U, P_{de}, P_{in})$ feasible to* (19),(21), (22) *will not decrease the objective in **P2** compared with that of the equilibrium.*

Definition 1 states that all the players choose the optimal decisions given the other's strategy at the equilibrium. The following theorem shows the property of the DR equilibrium.

Theorem 1. *There exists an equilibrium $(z^{U\text{-}G*}, \mathbf{x}^{U*}, P^*_{de}, P^*_{in})$, $(z^{D\text{-}U*}, P_i^{dp*}, k_i^*, i \in \mathcal{N})$, and $(\mathbf{p}^*, \mathbf{D}^{Ad*})$. In addition, any equilibrium is efficient, i.e., $(z^{D\text{-}U*}, \mathbf{p}^*, \mathbf{D}^{Ad*})$ is the unique optimum to **P3-D** and $(z^{U\text{-}G*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{Ad*})$ is the unique optimum to **P3-U**.*

Remarks: Theorem 1 shows the existence and efficiency of the DR game equilibrium. It states that the social welfare at the equilibrium (optimal value of **P1** and **P2**) is the same as the one under the coordinated setting (optimal value of **P3-D** and **P3-U**). It is worth noticing that such an equilibrium may not be unique. For example, if the equilibrium response $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ under some incentive price $P_{\text{de}} > 0$ are at their upper bounds, increasing (resp decreasing) the value of P_{de} (resp P_i^{dp}) will hence not influence the choice of $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ from Lemma 5, indicating the non-uniqueness of the equilibrium in the choice of optimal incentive/dynamic prices. Nevertheless, as the equilibrium is always efficient, the optimal DR solution of power supply and load adjustments $(z^{\text{U-G}*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ is unique from Lemma 4. In addition, Theorem 1 indicates that without directly solving the non-convex three-level game optimization **P2**, we can solve the convex **P3-U** instead to derive the unique optimal equilibrium DR decision that maximizes social welfare.

5.2 Equilibrium Incentive and Dynamic Pricing

5.2.1 Stage I: Incentive Price Offering of Utility. We further investigate the optimal incentive prices with which the optimal solutions of **P1** and **P2** maximize social welfare as shown in Theorem 2.

Theorem 2. *The optimum of **P1** $(\hat{\mathbf{p}}, \hat{\mathbf{D}}^{Ad})$ and the corresponding optimum $(\hat{z}^{U\text{-}G}, \hat{\mathbf{x}}^U)$ of **P2** are identical to the optimum of **P3-U** $(z^{U\text{-}G*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{Ad*})$ and **P3-D** if the incentive prices are set as:*

- *If $f'(z^{U\text{-}G*}) > P_{buy}$, we can set $P^*_{in} = 0$ and*

$$P^*_{de} = \begin{cases} f'(z^{U\text{-}G*}) - P_{sell}, & \text{if } z^{D\text{-}U*} < 0; \\ f'(z^{U\text{-}G*}) - P_{buy}, & \text{if } z^{D\text{-}U*} > 0; \\ f'(z^{U\text{-}G*}) - P_{s\text{-}b}, & \text{if } z^{D\text{-}U*} = 0, \end{cases} \tag{24}$$

for any $P_{s\text{-}b} \in [P_{sell}, P_{buy}]$.

- *If $f'(z^{U\text{-}G*}) < P_{sell}$, we can set $P^*_{de} = 0$ and*

$$P^*_{in} = \begin{cases} P_{sell} - f'(z^{U\text{-}G*}), & \text{if } z^{D\text{-}U*} < 0; \\ P_{buy} - f'(z^{U\text{-}G*}), & \text{if } z^{D\text{-}U*} > 0; \\ P_{s\text{-}b} - f'(z^{U\text{-}G*}), & \text{if } z^{D\text{-}U*} = 0. \end{cases} \tag{25}$$

for any $P_{s\text{-}b} \in [P_{sell}, P_{buy}]$.

- *If $P_{sell} \le f'(z^{U\text{-}G*}) \le P_{buy}$, we can set*

$$\begin{cases} P^*_{de} = f'(z^{U\text{-}G*}) - P_{sell}, P^*_{in} = 0, & \text{if } z^{D\text{-}U*} < 0; \\ P^*_{in} = P_{buy} - f'(z^{U\text{-}G*}), P^*_{de} = 0, & \text{if } z^{D\text{-}U*} > 0. \end{cases} \tag{26}$$

If $z^{D\text{-}U*} = 0$, suppose $f'(z^{U\text{-}G*}) = P^*_{s\text{-}b} \in [P_{sell}, P_{buy}]$, set

$$\begin{cases} P^*_{de} = f'(z^{U\text{-}G*}) - \tilde{P}_{s\text{-}b}, P^*_{in} = 0; \quad or \\ P^*_{in} = \hat{P}_{s\text{-}b} - f'(z^{U\text{-}G*}), P^*_{de} = 0, \end{cases} \tag{27}$$

for any $\tilde{P}_{s\text{-}b} \in [P_{sell}, P^*_{s\text{-}b}]$ or $\hat{P}_{s\text{-}b} \in [P^*_{s\text{-}b}, P_{buy}]$.

*Here $f'(z^{U\text{-}G*}) \in \partial f(z^{U\text{-}G*})$ is a subderivative of f at $z^{U\text{-}G*}$ that satisfies the KKT condition of **P3-U**.*

Remarks: Here $z^{\text{D-U}*} = \sum_{i\in\mathcal{N}} D_i^* - \sum_{i\in\mathcal{N}}(x_i^{D*} + p_i^* + D_i^{\text{Ad}^*})$ is calculated from (16). Theorem 2 provides the closed-form optimal incentive price-setting scheme to achieve the efficient equilibrium. It indicates that when utility is confronted with high (resp low) marginal price in the electricity market, i.e., larger (resp smaller) $f'(z^{\text{U-G}*})$, utility will encourage the DSO to decrease (resp increase) the energy usage with $P_{\text{de}} > 0$ (resp $P_{\text{in}} > 0$) respectively. It also suggests that the larger/smaller the marginal price, the larger optimal incentive price $P_{\text{de}}/P_{\text{in}}$. Therefore, the incentive price can be seen as the price signal passing from utility to DSO. It indicates the fluctuation of the marginal energy cost from the external grid and the unit monetary reward to DSO for the improvement in social welfare.

5.2.2 Stage II: Dynamic Pricing of DSO. In this subsection, we further provide the optimal dynamic prices of DSO to each prosumer.

Theorem 3. *The optimum of **DR-sub** $(\hat{\mathbf{p}}, \hat{\mathbf{D}}^{Ad})$ and the corresponding $\hat{z}^{D\text{-}U}$ are practical, i.e., $(\hat{z}^{D\text{-}U}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{Ad})$ is*

- *feasible to* (3), (12),(14),(16), (17) *and identical to the optimum in **P3-U** and **P3-D** and maximizes utility's social welfare B_{social} and DSO's benefit $\tilde{B}^D_{total}$;*
- *maximizing DSO's revenue, i.e., is the maximizer of*

$$\max_{z^{D\text{-}U}, \mathbf{p}, \mathbf{D}^{Ad}} \sum_{i\in\mathcal{N}} (P_i^{dp} \cdot D_i^{net} + k_i) - C_u(z^{D\text{-}U}) + R(z^{D\text{-}U})$$
$$\text{s.t.} \quad (3), (12), (14), (16), (17),$$
$$D_i^{net} = D_i^* - D_i^{Ad} - x_i^{D*} - p_i - e_i. \tag{28}$$

- *budget balance: the total payments received by DSO from prosumers is no more than the payment to utility, i.e., $\sum_{i\in\mathcal{N}}(P_i^{dp*} \cdot D_i^{net*} + k_i^*) \le C_u(z^{D\text{-}U*}) - R(z^{D\text{-}U*})$,*

[7] In case the utility and DSO are the same entity, the three-level stage game degenerates to the two-stage game. We remark that the desired equilibrium between utility/DSO and prosumers still exists by setting the optimal dynamic prices as Theorem 3 in Sec. 5.2

[8] The analysis holds for multiple utilities/DSOs and the equilibrium for each network can be obtained separately. We first focus on a single utility/DSO.

[9] Utility will only provide $(P_{\text{de}}, P_{\text{in}})$ to DSO and the corresponding optimal $(z^{\text{U-G}}, \mathbf{x}^U)$ is obtained by solving **P2** with fixed $(\mathbf{p}, \mathbf{D}^{\text{Ad}}) = \arg \mathbf{P1}|_{(P_{\text{de}}, P_{\text{in}})}$.

if the dynamic pricing for each prosumer $i \in \mathcal{N}$ is set as:

$$P_i^{dp} = -\eta - \sum_{l=1}^{E} (\bar{\epsilon}_l a^{(l,i)} - \underline{\epsilon}_l a^{(l,i)}), \tag{29}$$

$$k_i = \frac{\sum_{j \neq i} \left(P_I D_j^{net*} + (P_{in}^* - P_{de}^*) \cdot (D_j^{Ad*} + p_j^*) - P_j^{dp} D_j^{net*} \right)}{N-1}. \tag{30}$$

*Here $P_I = P_{buy}$ if $z^{D\text{-}U*} \geq 0$ and $P_I = P_{sell}$ if $z^{D\text{-}U*} < 0$. The net demand $D_i^{net*} = D_i^* - D_i^{Ad*} - x_i^{D*} - p_i^* - e_i$. $(p_i^*, D_i^{Ad*}, z^{D\text{-}U*} = \sum_{i\in\mathcal{N}} D_i^{net*})$ and $(\eta, \bar{\epsilon}_l, \underline{\epsilon}_l)$ are the optimal solutions and the corresponding Lagrangian dual variables of the KKT conditions for **P3-U** or **P3-D** as discussed in Appendix C respectively.*

Theorem 3 provides the closed-form dynamic price-setting for each prosumer. Under such a setting, given any $(P_{\text{de}}, P_{\text{in}})$, DSO's objective, i.e., the regional aggregated benefit, is maximized and prosumers' actions at the DSO-prosumers game-theoretical setting are the same as the optimal ones under the coordinated setting. Here P_i^{dp} can be interpreted as the *Locational Marginal Price* as discussed in Sec. 3.2.4. Together with Theorem 1, we summarize that the equilibrium load adjustment (prosumers' DR decision corresponding to the Stage III problem in **DR-sub**) can be obtained by the optimum $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ of **P3-U**. The utility's optimal equilibrium incentive price-setting scheme is then given by Theorem 2 and DSO's dynamic price-setting scheme can be determined by Theorem 3, which is the solution of Stage I utility's problem in **P1** and Stage II DSO's problem in **P2**. Therefore, the three-stage multi-level equilibrium can be recovered and reached directly by solving a convex problem **P3-U**. The problem structure and decomposition are given in Fig. 3.

As seen from Theorem 2 and Theorem 3, the equilibrium of the DR game depends on the optimal solution and the corresponding dual variables of **P3-U**, in which the different levels of restrictiveness of constraints will influence the corresponding equilibrium outcome. We have the following corollary stating the continuity of the optimal DR decisions w.r.t. the problem parameters.

Corollary 4. *The equilibrium DR decision $(z^{U\text{-}G*}, \mathbf{x}^{U*}, z^{D\text{-}U*}, \mathbf{p}^*, \mathbf{D}^{Ad*})$ is continuous w.r.t. the uncontrollable $(\mathbf{r}, \mathbf{U})$ and the constraints limits $(\mathbf{L}, \bar{\mathbf{x}}^D, \underline{\mathbf{x}}^D, \bar{\mathbf{D}}, \underline{\mathbf{D}}, \bar{\mathbf{x}}^U, \underline{\mathbf{x}}^U, z_{\min}^{D\text{-}U}, z_{\max}^{D\text{-}U})$.*

The continuity of $(z^{\text{U-G}*}, \mathbf{x}^{U*}, z^{\text{D-U}*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ comes from the uniqueness of the optimal solution of **P3-D**/**P3-U** [5, 29] while may not smooth.[10] Due to the non-uniqueness of the optimal dual variables in Theorem 3 and the price-setting scheme in Theorem 2, the optimal $(P_{\text{de}}^*, P_{\text{in}}^*, P_i^{dp*}, k_i^*, i \in \mathcal{N})$ may not be continuous. We leave the analysis of the dis-continuous/unsmooth of the DR equilibrium under general AC-PF constraints for future work.

5.3 Robustness-Enhanced Equilibrium

We further provide a robustness-enhanced design based on the existing scheme against *false* information from DSO and prosumers. We first have the following observation. Profile $(\mathbf{x}^{D*} + \mathbf{p}^*, \mathbf{D}^* - \mathbf{D}^{\text{Ad}*}, z^{\text{U-G}*}, \mathbf{x}^{U*})$ is the overall optimal energy consumption and generation in the power network based on the game-theoretical setting. Note that (12),(14),(17) is identical to (5),(7),(2) by treating $(\mathbf{x}^{D*} + \mathbf{p}, \mathbf{D}^* - \mathbf{D}^{\text{Ad}})$ as global variables, indicating that the optimum $(\mathbf{x}^{D*} + \mathbf{p}^*, \mathbf{D}^* - \mathbf{D}^{\text{Ad}*}, z^{\text{U-G}*}, \mathbf{x}^{U*})$ is irrelevant to the specific choice of $(\mathbf{x}^{D*}, \mathbf{D}^*, z^{\text{D-U}*})$. In addition, the optimal incentive price-setting in Theorem 2 and dynamic price-setting in Theorem 3 are only related to $(z^{\text{U-G}*}, x^{U*})$[11] and the dual variables. This implies that even the DSO/prosumers report the *false* $(z_*^{\text{D-U}} = \sum_{i\in\mathcal{N}}(D_i^* - x_i^* - e_i))$, by solving **P3-U**, the utility (resp DSO) can still get the *correct* incentive (resp dynamic) prices to its follower. Based on the above observation, we modify DSO's reward function in (18) as

$$R(z^{\text{D-U}}) = P_{\text{de}} \cdot (z_*^{\text{D-U}} - z^{\text{D-U}}) + P_{\text{in}} \cdot (z^{\text{D-U}} - z_*^{\text{D-U}}). \tag{31}$$

In the above DR reward design, if utility provides a $(P_{\text{de}} > 0, P_{\text{in}} = 0)$ (resp $(P_{\text{de}} = 0, P_{\text{in}} > 0)$) and DSO's net load adjustment $(z_*^{\text{D-U}} - z^{\text{D-U}}) < 0$ (resp $(z_*^{\text{D-U}} - z^{\text{D-U}}) > 0$) under its (false) report, DSO will be *penalized*. Recall that if DSO reports the truthful $z_*^{\text{D-U}}$, its reward will always be non-negative according to Lemma 5 as its optimal net adjustment is non-negative (resp non-positive) given $P_{\text{de}} > 0$ (resp $P_{\text{in}} > 0$). In addition, though DSO may exploit additional benefits by reporting less/more net load than real load, e.g., reporting more load and utility provides a $P_{\text{de}} > 0$, its worst case benefit can be decreased if utility provides a $P_{\text{in}} > 0$ (resp $P_{\text{de}} > 0$) while it reports more (resp less) net demand. We have the following theorem.

Theorem 5. *Any equilibrium with (31) is robust to DSO and prosumers' pre-DR reported $(z_*^{D\text{-}U}, \mathbf{x}^{D*}, \mathbf{D}^*)$. That is, the optimal incentive and dynamic prices (P_{de}^*, P_{in}^*) and $(P_i^{dp*}, k_i^*, i \in \mathcal{N})$ and the optimal energy profile $(\mathbf{x}^{D*} + \mathbf{p}^*, \mathbf{D}^* - \mathbf{D}^{Ad*}, z^{U\text{-}G*}, \mathbf{x}^{U*})$ that maximize social welfare are irreverent to $(z_*^{D\text{-}U}, \mathbf{x}^{D*}, \mathbf{D}^*)$. In addition, DSO will always report truthfully to maximize its worst-case benefit under arbitrary prices f'.*

Remark: Theorem 5 shows the desirable robustness property of the equilibrium. It indicates that whatever (fault information) $(z_*^{\text{D-U}}, \mathbf{x}^{D*}, \mathbf{D}^*)$ DSO/prosumers reported to utility/DSO, the resulting power network energy consumption, generation, and energy trading are optimal such that social welfare is still maximized and the corresponding incentive/dynamic prices under such a fault-ridden setting are the same as the optimal ones. Furthermore, to maximize the worst-case benefit under arbitrary grid prices f',

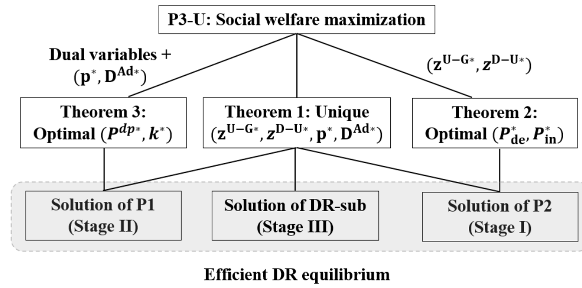


Figure 3: Structure of the problem decomposition to derive the DR equilibrium.

[10]The smoothness of the optimum depends on the choice of the objective function, e.g., piece-wise linear if the objective is quadratic [12]. The optimum may not smooth for the general convex objective function.

[11] $z^{\text{D-U}*} = z^{\text{U-G}*} + \sum_{j\in\mathcal{M}} x_j^{U*}$ from (16) and (21).

DSO will always report the truthful $z_*^{\text{D-U}} = \sum_{i\in\mathcal{N}}(D_i^* - x_i^* - e_i)$. In addition, given the equilibrium dynamic pricing in Theorem 3, prosumers will always choose the optimal equilibrium DR decision otherwise they will suffer lower benefits. We leave the design of strictly enforcing prosumers' behaviors in following the agreement and investigating the impact of imprecise modeling mistakes for future work.

5.4 Impact of Renewable and Uncontrollable Load Uncertainty

In practice, local renewable generations can be non-dispatchable due to their intermittent and unpredictable nature. Uncontrollable loads can also present uncertainty in appliances' daliy usage. In this subsection, we further explore the impact of renewable/uncontrollable load uncertainty on the objectives of prosumers, DSO, and utility. Assume the net uncontrollable power injection $e_i, i \in \mathcal{N}$ follows a distribution within a certain range. We use variance σ_i^2 to represent its fluctuation and uncertainty. We have the following results.

THEOREM 6. *The larger renewable uncertainty will not cause more benefits. In particular,*

- *utility's expected benefit (social welfare) on B_{social} is not increasing w.r.t. $\sigma_i^2, i \in \mathcal{N}$;*
- *given (P_{de}, P_{in}) and assume the reward function is given as* (31)*, DSO's expected benefits on B_{total}^D and $\tilde{B}_{total}^D$ are not increasing w.r.t. $\sigma_i^2, i \in \mathcal{N}$;*
- *for each $i \in \mathcal{N}$, given (P_i^{dp}, k_i), prosumer i's expected benefit on $B_{pro,i}$ and $B_{pro,i}^{DR}$ is not increasing w.r.t. $\sigma_i^2, i \in \mathcal{N}$.*

Theorem 6 states that a larger uncertainty on e_i will decrease the DR participants' benefits. As a result, they will have the incentive to increase the renewable/uncontrollable load prediction accuracy for larger expected benefits. See Fig. 10 for an illustration. In addition, if r_i and U_i are non-negatively correlated, the increase of the variance of r_i and U_i will lead a larger σ_i, introducing the lower benefits.

6 DISTRIBUTED ALGORITHM

Based on the observation from Theorem 1 to Theorem 3, we develop an efficient distributed algorithm to solve **P3-U** cooperatively by utility and DSO/prosumers. The algorithm demonstrates the interactions among DR participants when the proposed DR scheme is employed in real-world smart grid systems. Consider the following reformulated **P3-R** of **P3-U** by substituting equality (21) as follows:

P3-R: Reformulated Social Welfare Maximization

$$\max \; -f\Big(\sum_{i\in\mathcal{N}} D_i^* - \sum_{j\in\mathcal{M}} x_j^U - \sum_{i\in\mathcal{N}}(x_i^{D*} + p_i + D_i^{Ad} + e_i)\Big) + \sum_{i\in\mathcal{N}}(\tilde{B}_i(D_i^{\text{Ad}}) - \tilde{C}_{g,i}^D(p_i)) - \sum_{j\in\mathcal{M}} C_{g,j}^U(x_j^U), \tag{32}$$

s.t. (12), (14), (17), (19),

$$z_{\min}^{\text{D-U}} \le \sum_{i\in\mathcal{N}} D_i^* - \sum_{i\in\mathcal{N}}(x_i^{D*} + p_i + D_i^{Ad} + e_i) \le z_{\max}^{\text{D-U}}, \tag{33}$$

var. $\mathbf{x}^U, \mathbf{p}, \mathbf{D}^{\text{Ad}}$.

Algorithm 1 Distributed Projected Gradient Descent Method

1: **Input:** Step sizes t_u, t_d, initial $((\mathbf{x}^U)^0, (\mathbf{p})^0, (\mathbf{D}^{\text{Ad}})^0)$, initial $(f')^0$, **P0** net load $\sum_i(D_i^* - x_i^{D*} - e_i)$, tolerance χ, $k = 0$
2: **Output:** Optimal solution of **P3-R**
3: **repeat**
4: Utility updates $(\mathbf{x}^U)^{k+1}$ via (34)
5: DSO/prosumers updates $((\mathbf{p})^{k+1}, (\mathbf{D}^{\text{Ad}})^{k+1})$ via (35)
6: DSO passes $\sum_i((p)_i^{k+1} + (D^{\text{Ad}})_i^{k+1})$ to utility
7: Utility updates $(f')^{k+1}$ and pass it to DSO
8: $k = k + 1$
9: **until** $||(\mathbf{x}^U)^{k+1} - (\mathbf{x}^U)^k|| \le \chi$, and $||((\mathbf{p})^{k+1}, (\mathbf{D}^{\text{Ad}})^{k+1}) - ((\mathbf{p})^k, (\mathbf{D}^{\text{Ad}})^k)|| \le \chi$
10: **return** $(\mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*}) := ((\mathbf{x}^U)^{k+1}, (\mathbf{p})^{k+1}, (\mathbf{D}^{\text{Ad}})^{k+1})$

(33) is the energy trading limit between DSO and utility reformulated from (3) and (21). Note that the variables of utility ($\mathbf{x}^U$) and DSO/Prosumers ($\mathbf{p}, \mathbf{D}^{\text{Ad}}$) are separable in terms of the constraints in **P3-R**. We hence apply the *Projected Gradient Descent Method* [28] to solve **P3-R** in a distributed manner. In each iteration, utility updates its generation $\mathbf{x}^U$ and DSO updates the local generation and demand adjustments $(\mathbf{p}, \mathbf{D}^{\text{Ad}})$:[12]

- Utility updates $\mathbf{x}^U$:

$$\begin{cases} (\tilde{x}_j^U)^{k+1} = (x_j^U)^k + t_u\left((f')^k - \dfrac{dC_{g,j}^U}{d(x_j^U)^k}\right), \\ (\mathbf{x}^U)^{k+1} = \pi_{C_1}[(\tilde{\mathbf{x}}^U)^{k+1}]. \end{cases} \tag{34}$$

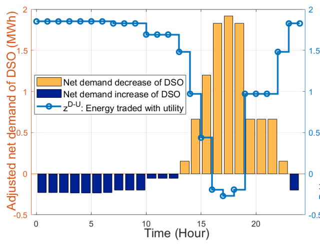
- DSO/Prosumers update $\mathbf{p}, \mathbf{D}^{\text{Ad}}$:

$$\begin{cases} (\tilde{p}_i)^{k+1} = (p_i)^k + t_d\left((f')^k - \dfrac{d\tilde{C}_{g,i}^D}{d(p_i)^k}\right), \\ (\tilde{D}_i^{\text{Ad}})^{k+1} = (D_i^{\text{Ad}})^k + t_d\left((f')^k + \dfrac{d\tilde{B}_i}{d(D_i^{\text{Ad}})^k}\right), \\ \left[(\mathbf{p})^{k+1}, (\mathbf{D}^{\text{Ad}})^{k+1}\right] = \pi_{C_2}\left[(\tilde{\mathbf{p}})^{k+1}, (\tilde{\mathbf{D}}^{\text{Ad}})^{k+1}\right]. \end{cases} \tag{35}$$

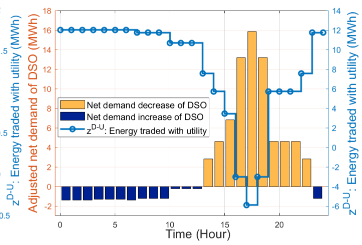
Here k is the iteration number, and t_u, t_d are the step sizes. π_{C_1} and π_{C_2} denote the projection operations onto sets $C_1 = \{\mathbf{x}^U|(19) \text{ holds}\}$ and $C_2 = \{\mathbf{p}, \mathbf{D}^{\text{Ad}}|(12), (14), (17), (33) \text{ hold}\}$. For convex optimizations with the unique optimal solution, the projected gradient descent method will converge to the unique optimum for sufficiently small step sizes t_u and t_d [16, 28].

Details of the implementation are given in Algorithm 1. In each iteration, utility and DSO/prosumers update their decision variables $\mathbf{x}^U$ and $(\mathbf{p}, \mathbf{D}^{\text{Ad}})$ locally (Line 4-5) since sets C_1 and C_2 only depend on their own decision variables. DSO then passes the aggregate updated net adjustment $\sum_{i\in\mathcal{N}}((p)_i^{k+1} + (D^{\text{Ad}})_i^{k+1})$ to utility through the communication network (Line 6) to further calculate the gradient of f or to verify whether the stopping criterion is reached (Line 10). The updated f' is then sent back to DSO by utility (Line 7). We remark that in practice, utility (resp DSO) do not need to get the specific value of each $(\mathbf{x}^{D*}, \mathbf{D}^*)$ from **P0** and $(\mathbf{p}, \mathbf{D}^{\text{Ad}})$ (resp $\mathbf{x}^U$) at each iteration. Accessing only $\sum_{i\in\mathcal{N}}(D_i^* - x_i^{D*})$ and $\sum_{i\in\mathcal{N}}((p)_i^{k+1} + (D^{\text{Ad}})_i^{k+1})$ would provide the efficient and robust equilibrium from Algorithm 1 for better privacy preservation. We

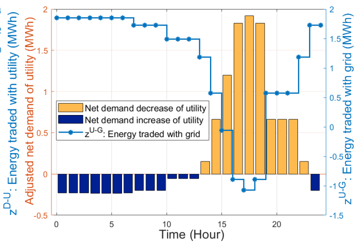
[12] We consider differentiable $f, C_{g,i}^D, C_{g,i}^D$ here.



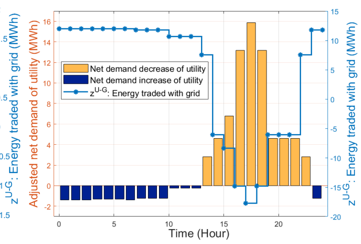
(a) DSO in Case30.



(b) DSO in Case118.



(c) Utility in Case30.



(d) Utility in Case118.

Figure 4: DSO's demand response and utility's net demand adjustment in Case30 and Case118

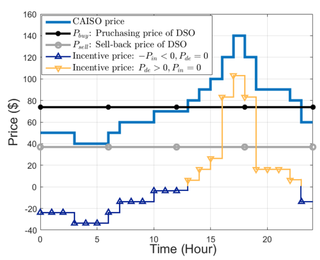


Figure 5: CAISO prices and optimal incentive prices.

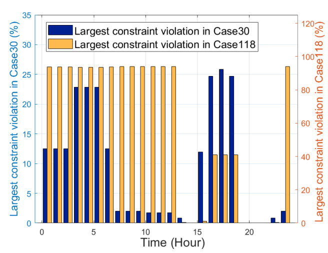
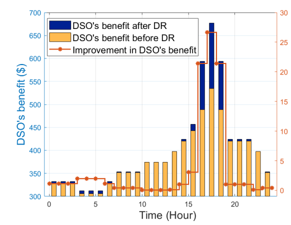
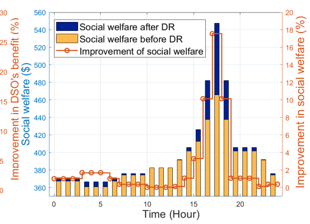


Figure 6: Maximum violation due to ignoring the operational constraints.

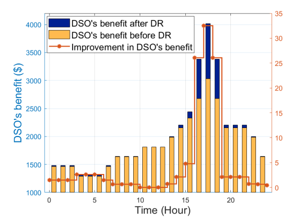


(a) DSO's benefit.

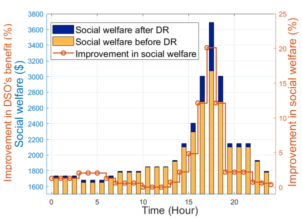


(b) Social welfare.

Figure 7: Improvements for DSO and social welfare in Case30.



(a) DSO's benefit.



(b) Social welfare.

Figure 8: Improvements for DSO and social welfare in Case118.

remark that in addition to the designed distributed projected gradient decent algorithm between utility and DSO to solve **P3-U**, there exist other distributed algorithms, e.g., primal-dual algorithm and alternative direction method of multipliers (ADMM) algorithm, to further include the participation of prosumers, which can also reach the desired equilibrium of the DR game cooperatively.

In conclusion, as shown in Fig. 2, after DSO solves **P0** to derive the pre-DR decision $(z_*^{\text{D-U}}, \mathbf{x}^{D*}, \mathbf{D}^*)$, the optimal equilibrium solution is obtained by solving **P3-R** in a distributed manner. The optimal incentive price is further derived by Theorem 2. In addition, DSO can further obtain the optimal dual variables of **P3-D**/**P3-U** with the optimal solution through a set of linear KKT conditions system. The optimal dynamic price to each prosumer can then be determined from Theorem 3.[13] Together with the **P3-R** solution, the efficient and robust DR game equilibrium can hence be constructed.

7 PERFORMANCE EVALUATION

In this section, we provide numerical evaluations on the proposed DR scheme considering different numbers of prosumers interconnected in the IEEE 30/118-bus test systems [1]. Each prosumer can be equipped with its local generator, and the node where the utility is located is seen as the slack bus. We use the time of use (ToU) purchasing price in California Independent System Operator (CAISO) [36] as the per-unit cost for utility purchasing power from the grid as shown in Fig. 5. The buying price of DSO (P_{buy}) is set as the 24 hour average of the CAISO prices and $P_{\text{sell}} = 0.5 \times P_{\text{buy}}$ to make it consistent with (4) [19].

[13]Given the incentive and dynamic prices, DSO and prosumers will always choose the equilibrium solution as their best responses as shown in Theorem 2 and Theorem 3.

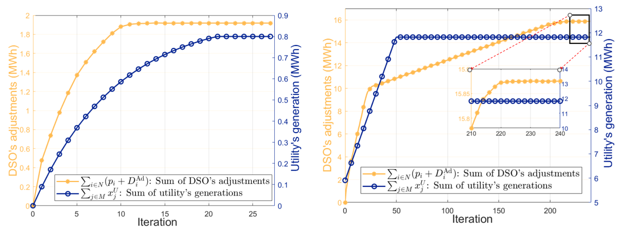
7.1 Improvement of the Proposed DR Scheme

As shown in Fig. 5, utility will provide $P_{\text{de}} > 0, P_{\text{in}} = 0$ in peak hours, e.g., from 14.p.m-23.p.m, and $P_{\text{de}} = 0, P_{\text{in}} > 0$ in valley hours according to the incentive price-setting scheme in Theorem 2. It indicates that utility prefers to decrease (resp increase) the imported energy from the external grid by inducing DSO and prosumers to decrease (resp increase) energy usage when the wholesale grid price is high (resp low) to maximize the network social welfare.

Fig. 4 shows the adjusted demands and the net energy trading of DSO and utility in 24 hours. We observe that in peak hours, the net demand is decreased significantly with the larger incentive prices $P_{\text{de}} > 0$ provided, e.g., 116.5% of DSO and 226.4% of utility in Case30 and 159.3% and 855.6% in Case118, resulting in the net exports from

Table 2: Performance improvement of the proposed DR scheme.

Metric		Case30 (%)	Case118 (%)
Social welfare	peak hour	17.6	20.1
	average	2.6	3.9
DSO's benefit	peak hour	26.6	32.5
	average	4.6	6.5
Constraints violation	consider limits	0	0
	ignore limits	25.8	94.0



(a) Case30. (b) Case118.

Figure 9: Convergence of the projected gradient descent algorithm with $\chi = 10^{-6}$.

DSO to utility and from utility to grid to exploit the high reward and wholesale marginal prices. Similarly, when the grid price is low in valley hours, they will increase energy usage and decrease local generations to exploit the cheap external grid power with the positive $P_{\text{in}} > 0$ provided for larger benefits.

The social welfare improvement is measured as the difference between the optimal objective of **P3-U** and the objective of **P2** with $(\mathbf{p} = \mathbf{0}, \mathbf{D}^{\text{Ad}} = \mathbf{0})$, indicating the absence of DR. As seen in Fig. 7 and Fig. 8, the social welfare is significantly improved, especially in peak hours by up to 17.6% and 2.6% on average in Case30 and 20.1% and 3.9% in Case118. Meanwhile, the increase of DSO's benefit is up to 26.6% and 4.6% on average in Case30 and 32.5% and 6.5% in Case118. The results are listed in Table 2.

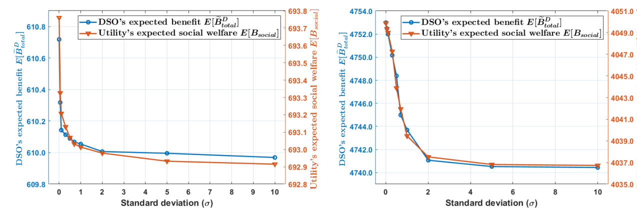
Next, we present the necessity of considering power network constraints in Fig. 6. It is shown that without considering the operational constraints, the resulting DR solution could cause severe violations and hence lead the design infeasible, e.g., 25.8% and 94.0% maximum violation in Case30 and Case118 respectively.[14]

We further present the convergence of the developed projection-based distributed algorithm to solve **P3-R** in Fig. 9. Results demonstrate the effectiveness of Algorithm 1 with the solution convergence in 28/239 iterations for Case30/Case118.

7.2 Renewable Uncertainty Evaluation

In this subsection, we further present the simulation results on the impact of renewable uncertainty. Fig. 10 demonstrates that the expected social welfare and DSO's benefit are decreasing w.r.t. the increase of net uncontrollable power injection variance. In our simulation, we consider the renewable capacities at the generation loads with capacities of 20% to the corresponding node generation limits. We model the renewable uncertainty as the *truncated normal distributions* between 0 and 40% of local generation limit with the variance increases from 0 to 100. We observe that the network social welfare decreases from 610.7 to 610.0 for Case30 and from 4753.0 to 4740.4 for Case118, which coincides with Theorem 6. Similar trends hold for DSO's benefit as shown in Fig. 10.

[14]The maximum constraint violation is measured as the largest percentage violation over the limit among all constraints in the power network.



(a) Case30. (b) Case118.

Figure 10: The expected benefits decrease with the increasing renewable uncertainty.

8 CONCLUSION AND FUTURE DIRECTIONS

We develop a novel demand response mechanism between utility, DSO, and prosumers based on the three-stage multi-level game model considering power system operational constraints. In DR, utility offers incentive prices to DSO and in response, DSO provides dynamic prices to prosumers to guide them in determining load adjustments. We show (i) the DR game admits an efficient equilibrium that maximizes social welfare and DSO/prosumers' benefits while respecting all system constraints and (ii) the uniqueness of optimal utility's power supply and prosumers' demand/generation adjustments. (iii) We provide the optimal incentive/dynamic price-setting schemes to achieve the equilibrium, (iv) a robustness-enhanced design against DSO's fault information, (v) and investigate the impact of renewable/uncontrollable loads uncertainty. (vi) We further develop an efficient distributed algorithm to help utility and DSO reach the equilibrium jointly. Simulation results show the superior performance of the proposed scheme in improving social welfare by 20.1% and DSO's benefit by 32.5% while respecting all system constraints. Note that the projection problems need to be solved repeatedly in the design. Our preliminary results show that applying Deep Learning techniques can achieve desirable convergence speedups to tackle the load/renewable uncertainty challenges, which we leave for future work. Another compelling future direction is to apply the proposed DR scheme to the more comprehensive full AC-PF setting with time-coupling energy storage systems, which incorporates the non-convex line transfer loss and the voltage constraints in the power network.

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A PROOF OF LEMMA 1, LEMMA 5, AND LEMMA 4

A.1 Proof of Lemma 1

Note that **P0**'s objective is strictly concave w.r.t. (x_i^D, D_i) and piecewise linear concave w.r.t. $z^{\text{D-U}}$. Suppose **P0** has two different optimums $(\hat{z}^{\text{D-U}}, \hat{\mathbf{x}}^D, \hat{\mathbf{D}})$ and $(\tilde{z}^{\text{D-U}}, \tilde{\mathbf{x}}^D, \tilde{\mathbf{D}})$ with the same objective value. If $(\hat{\mathbf{x}}^D, \hat{\mathbf{D}}) \neq (\tilde{\mathbf{x}}^D, \tilde{\mathbf{D}})$, we can construct a feasible point $(z_*^{\text{D-U}}, \mathbf{x}^{D*}, \mathbf{D}^*) = \alpha\cdot(\hat{z}^{\text{D-U}}, \hat{\mathbf{x}}^D, \hat{\mathbf{D}}) + (1-\alpha)\cdot(\tilde{z}^{\text{D-U}}, \tilde{\mathbf{x}}^D, \tilde{\mathbf{D}})$ as the constraints are all linear such that the objective is strictly larger than the optimal one $\forall\alpha \in (0,1)$, which creates contradiction. Therefore, we have $(\hat{\mathbf{x}}^D, \hat{\mathbf{D}}) = (\tilde{\mathbf{x}}^D, \tilde{\mathbf{D}})$ and from (1) $\hat{z}^{\text{D-U}} = \tilde{z}^{\text{D-U}}$. This completes the proof.

A.2 Proof of Lemma 5

We prove Lemma 5 by contradiction. Consider **P3-D** with $P_{\text{de}} > 0, P_{\text{in}} = 0$, denote its optimal solution as $(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}})$ and $\tilde{B}^D_{\text{total}} = \sum_{i\in\mathcal{N}}(\tilde{B}_d(D_i^{\text{Ad}}) - \tilde{C}_g(p_i)) - C_u(z^{\text{D-U}})$ as $B(z^{\text{D-U}}, \mathbf{p}, \mathbf{D}^{\text{Ad}})$. Note that $(z_*^{\text{D-U}} - z^{\text{D-U}}) = \sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad})$. Suppose $\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}) < 0$, we have

$$B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}) + P_{\text{de}} \cdot \max(\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}), 0) \tag{36}$$

$$= B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}) < B(z^{\text{D-U}} = z_*^{\text{D-U}}, \mathbf{p} = \mathbf{0}, \mathbf{D}^{\text{Ad}} = \mathbf{0}). \tag{37}$$

The last inequality holds as $(z^{\text{D-U}} = z_*^{\text{D-U}}, \mathbf{p} = \mathbf{0}, \mathbf{D}^{\text{Ad}} = \mathbf{0})$ is unique optimal to **P0** (Lemma 1). Hence, any $(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}})$ with $\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}) < 0$ can not be optimal to **P3-D**.

As to the monotonicity, consider two prices $P^1_{\text{de}} > P^2_{\text{de}}$ with optimums $(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}})$ and $(\tilde{z}^{\text{D-U}}, \tilde{\mathbf{p}}, \tilde{\mathbf{D}}^{\text{Ad}})$ such that $\sum_{i\in\mathcal{N}}(\tilde{p}_i + \tilde{D}_i^{\text{Ad}}) \geq 0$ and $\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{\text{Ad}}) \geq 0$, we have

$$\begin{cases} B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}) + P^1_{\text{de}} \cdot \sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}) \\ \geq B(\tilde{z}^{\text{D-U}}, \tilde{\mathbf{p}}, \tilde{\mathbf{D}}^{\text{Ad}}) + P^1_{\text{de}} \cdot \sum_{i\in\mathcal{N}}(\tilde{p}_i + \tilde{D}_i^{Ad}). \end{cases} \tag{38}$$

$$\begin{cases} B(\tilde{z}^{\text{D-U}}, \tilde{\mathbf{p}}, \tilde{\mathbf{D}}^{\text{Ad}}) + P^2_{\text{de}} \cdot \sum_{i\in\mathcal{N}}(\tilde{p}_i + \tilde{D}_i^{Ad}) \\ \geq B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}) + P^2_{\text{de}} \cdot \sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}). \end{cases} \tag{39}$$

Combining (38) and (39) gives $\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}) \geq \sum_{i\in\mathcal{N}}(\tilde{p}_i + \tilde{D}_i^{Ad})$. Similar results hold for $P_{\text{de}} = 0, P_{\text{in}} > 0$. The monotonicity of the optimal objective is self-evident.

As to the unique solution to **P3-D**, we know that **P3-D**'s optimum has $\sum_{i\in\mathcal{N}}(\hat{p}_i + \hat{D}_i^{Ad}) \geq 0$ (≤ 0) if $P_{\text{de}} > 0$ $(P_{\text{in}} > 0)$. As the objective is strictly concave w.r.t. $(\mathbf{p}, \mathbf{D}^{\text{Ad}})$ and concave w.r.t. $z^{\text{D-U}}$, one can prove the uniqueness similar to Appendix A.1.

A.3 Proof of Lemma 4

We sketch the proof here. As the objective in **P3-U** is strictly concave w.r.t. $(\mathbf{x}^U, \mathbf{p}, \mathbf{D}^{\text{Ad}})$ and f is convex w.r.t. $z^{\text{U-G}}$, similar to Appendix A.1, we can prove that **P3-U** has a unique optimum $(\mathbf{x}^U, \mathbf{p}^*, \mathbf{D}^{\text{Ad}*})$ and the optimal $z^{\text{U-G}*}$ is obtained from (21).

B PROOF OF LEMMA 2 AND LEMMA 3

B.1 Proof of Lemma 2

Consider the KKT conditions of **P0**. Primal feasibility is (1)-(3), (5), (7). Complementary slackness is constructed by the dual variables and the associated inequality constraints. We define the following dual variables for each constraint. η_0 is the dual variable of (1). $\bar{\zeta}_0$,$\underline{\zeta}_0$ are the dual variables of the upper and lower bound constraints in (3). $\bar{\theta}_{i,0}$, $\underline{\theta}_{i,0}$ and $\bar{\tau}_{i,0}$, $\underline{\tau}_{i,0}$ are the dual variables of (5) and (7). $\bar{\epsilon}_{l,0}$ and $\underline{\epsilon}_{l,0}$ are the dual variables of the branch flow limits in (2). The stationarity condition is:

- w.r.t. $z^{\text{D-U}}$, x_i^D and D_i, $i \in \mathcal{N}$:

$$\begin{cases} \partial[P_{\text{buy}} \cdot \max(z^{\text{D-U}}, 0) + P_{\text{sell}} \cdot \min(z^{\text{D-U}}, 0)] + \eta_0 + \bar{\zeta}_0 - \underline{\zeta}_0 \ni 0, \\ \partial C^D_{g,i}(x_i^D) + \eta_0 + \sum_{l=1}^E (\bar{\epsilon}_{l,0} a^{(l,i)} - \underline{\epsilon}_{l,0} a^{(l,i)}) + \bar{\theta}_{i,0} - \underline{\theta}_{i,0} \ni 0, \\ -\partial B_i(D_i) - \eta_0 - \sum_{l=1}^E (\bar{\epsilon}_{l,0} a^{(l,i)} - \underline{\epsilon}_{l,0} a^{(l,i)}) + \bar{\tau}_{i,0} - \underline{\tau}_{i,0} \ni 0. \end{cases} \tag{40}$$

Consider the KKT conditions of **P0-sub** then. Primal feasibility is (5), (7). Complementary slackness is constructed by the dual variables and the associated inequality constraints. Let $\bar{\varsigma}_{i,0}$, $\underline{\varsigma}_{i,0}$ and $\bar{\varkappa}_{i,0}$, $\underline{\varkappa}_{i,0}$ be the dual variables of (5), and (7). The stationarity condition is:

- w.r.t. x_i^D and D_i, $i \in \mathcal{N}$:

$$\begin{cases} \partial C^D_{g,i}(x_i^D) - P_i^{dp} + \bar{\varsigma}_{i,0} - \underline{\varsigma}_{i,0} \ni 0, \\ -\partial B_i(D_i) + P_i^{dp} + \bar{\varkappa}_{i,0} - \underline{\varkappa}_{i,0} \ni 0, \end{cases} \tag{41}$$

Note that if the dynamic pricing scheme is set as Lemma 2, one can easily check that the optimal solution of **P0** is also optimal to **P0-sub** by setting $(\bar{\varsigma}_{i,0}, \underline{\varsigma}_{i,0}, \bar{\varkappa}_{i,0}, \underline{\varkappa}_{i,0}) = (\bar{\theta}_{i,0}, \underline{\theta}_{i,0}, \bar{\tau}_{i,0}, \underline{\tau}_{i,0})$, i.e., the solution of **P0** satisfying the KKT conditions of **P0-sub**. Towards the second claim, one can see that the corresponding KKT conditions of the concerned program is given as

- w.r.t. $z^{\text{D-U}}$, x_i^D and D_i, $i \in \mathcal{N}$:

$$\begin{cases} \partial[P_{\text{buy}} \cdot \max(z^{\text{D-U}}, 0) + P_{\text{sell}} \cdot \min(z^{\text{D-U}}, 0)] + \eta_0' + \bar{\zeta}_0' - \underline{\zeta}_0' \ni 0, \\ P_i^{dp} + \eta_0' + \sum_{l=1}^E (\bar{\epsilon}'_{l,0} a^{(l,i)} - \underline{\epsilon}'_{l,0} a^{(l,i)}) + \bar{\theta}'_{i,0} - \underline{\theta}'_{i,0} \ni 0, \\ -P_i^{dp} - \eta_0' - \sum_{l=1}^E (\bar{\epsilon}'_{l,0} a^{(l,i)} - \underline{\epsilon}'_{l,0} a^{(l,i)}) + \bar{\tau}'_{i,0} - \underline{\tau}'_{i,0} \ni 0. \end{cases} \tag{42}$$

Therefore, we observe that the optimal solution of **P0** and **P0-sub** satisfies the above KKT conditions by setting $(\eta_0', \bar{\epsilon}'_{l,0}, \underline{\epsilon}'_{l,0}, \bar{\zeta}_0', \underline{\zeta}_0') = (\eta_0, \bar{\epsilon}_{l,0}, \underline{\epsilon}_{l,0}, \bar{\zeta}_0, \underline{\zeta}_0)$ and $(\bar{\theta}'_{i,0}, \underline{\theta}'_{i,0}, \bar{\tau}'_{i,0}, \underline{\tau}'_{i,0}) = \mathbf{0}$. Therefore, DSO's revenue is maximized under the solution of **P0-sub**. The third claim of *budget balance* condition is self-evident.

B.2 Proof of Lemma 3

The convexity and the uniqueness of the solution of **DR-sub** are from the strictly convex objective function w.r.t. (D_i^{Ad}, p_i) and the linear constraints. Consider the KKT conditions of **DR-sub**

- w.r.t. p_i and D_i^{Ad}, $i \in \mathcal{N}$:

$$\begin{cases} \partial\tilde{C}_{g,i}^D(p_i) - P_i^{dp} + \bar{\varsigma}_i - \underline{\varsigma}_i \ni 0, \\ -\partial\tilde{B}_i(D_i^{\text{Ad}}) - P_i^{dp} + \bar{\varkappa}_i - \underline{\varkappa}_i \ni 0. \end{cases} \tag{43}$$

For two different $P_i^{dp,1} > P_i^{dp,2}$, we must have

$$\tilde{C}_{g,i}^D(p_i^1) + \bar{\varsigma}_i^1 - \underline{\varsigma}_i^1 > \tilde{C}_{g,i}^D(p_i^2) + \bar{\varsigma}_i^2 - \underline{\varsigma}_i^2, \tag{44}$$

$$-\partial\tilde{B}_i(D_i^{\text{Ad},1}) + \bar{\varkappa}_i^1 - \underline{\varkappa}_i^1 > -\partial\tilde{B}_i(D_i^{\text{Ad},2}) + \bar{\varkappa}_i^2 - \underline{\varkappa}_i^2. \tag{45}$$

Since $\tilde{C}_{g,i}^D$ is a strictly convex and increasing function and $\partial\tilde{B}_i$ is a strictly concave and decreasing function, it is straightforward to obtain $p_i^1 \geq p_i^2$ and $D_i^{\text{Ad},1} \geq D_i^{\text{Ad},2}$. Here we use the superscript to denote the optimal decision variable p_i^* and $D_i^{\text{Ad}*}$ under different prices in Lemma 3 respectively.

C PROOF OF THEOREM 1, THEOREM 2, AND THEOREM 3

We proceed by investigating the Karush–Kuhn–Tucker (KKT) conditions of **P3-D** and **P3-U** respectively. Such conditions are both sufficient and necessary for convex optimizations with linear constraints (e.g., reformulated **P3-D-R** and **P3-U**) [20, 38].

C.1 KKT conditions for P3-U

Primal feasibility is (12),(14),(17),(19),(21),(23). Complementary slackness is constructed by the dual variables and the associated inequality constraints. We define the following dual variables for each constraint. η is the dual variable of (21). $\bar{\zeta}$,$\underline{\zeta}$ and $\bar{\theta}_j$, $\underline{\theta}_j$ are the dual variables of the upper and lower bound constraints in (23) and (19). $\bar{\epsilon}_l$ and $\underline{\epsilon}_l$ are the dual variables of the branch flow limits in (17). $\bar{\gamma}_i$, $\underline{\gamma}_i$ and $\bar{\tau}_i$, $\underline{\tau}_i$ are the dual variables of (12) and (14). The stationarity condition is:

- w.r.t. $z^{\text{U-G}}$, x_j^U, $j \in \mathcal{M}$, p_i and D_i^{Ad}, $i \in \mathcal{N}$:

$$\begin{cases} \partial f(z^{\text{U-G}}) + \eta + \bar{\zeta} - \underline{\zeta} \ni 0, \\ \partial C_{g,j}^U(x_j^U) + \eta + \bar{\zeta} - \underline{\zeta} + \bar{\theta}_j - \underline{\theta}_j \ni 0, \\ \partial\tilde{C}_{g,i}^D(p_i) + \eta + \sum_{l=1}^E(\bar{\epsilon}_l a^{(l,i)} - \underline{\epsilon}_l a^{(l,i)}) + \bar{\gamma}_i - \underline{\gamma}_i \ni 0, \\ -\partial\tilde{B}_i(D_i^{\text{Ad}}) + \eta + \sum_{l=1}^E(\bar{\epsilon}_l a^{(l,i)} - \underline{\epsilon}_l a^{(l,i)}) + \bar{\tau}_i - \underline{\tau}_i \ni 0, \end{cases} \tag{46}$$

We use f' to denote the subderivative such that $f'(z^{\text{U-G}}) + \eta + \bar{\zeta} - \underline{\zeta} = 0$ and $a^{(l,i)}$ is the (l,i)-th element of **PTDF**.

C.2 KKT conditions for P3-D

We first show that solving **P3-D** is equivalent to solving a reformulated version of it, called **P3-D-R**, by replacing (18) with (31) in the objective. Though **P3-D** is not convex, **P3-D-R** is convex with the unique optimum as its objective is (strictly) concave w.r.t. $\mathbf{p}, \mathbf{D}^{\text{Ad}}$, and $z^{\text{D-U}}$. Let $(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}})$ be the optimum to **P3-D-R** under some $P_{\text{de}} > 0$. We have

$$\begin{aligned} &B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}) + P_{\text{de}} \cdot \sum_{i \in \mathcal{N}} (\hat{p}_i + \hat{D}_i^{Ad}) \\ \geq &B(z^{\text{D-U}} = z_*^{\text{D-U}}, \mathbf{p} = \mathbf{0}, \mathbf{D}^{\text{Ad}} = \mathbf{0}). \end{aligned} \tag{47}$$

In addition, considering $P_{\text{de}} = 0$, we have

$$B(z^{\text{D-U}} = z_*^{\text{D-U}}, \mathbf{p} = \mathbf{0}, \mathbf{D}^{\text{Ad}} = \mathbf{0}) \geq B(\hat{z}^{\text{D-U}}, \hat{\mathbf{p}}, \hat{\mathbf{D}}^{\text{Ad}}). \tag{48}$$

Combining (47) and (48) gives $\sum_{i \in \mathcal{N}}(\hat{p}_i + \hat{D}_i^{\text{Ad}}) \geq 0$. Similar results hold for $P_{\text{in}} > 0$. Hence, the optimum of **P3-D-R** is also optimal to **P3-D** and vice versa, implying their equivalence.

We hence investigate the KKT conditions for **P3-D-R**. We use $\sum_{i \in \mathcal{N}} D_i^* - z^{\text{D-U}} - \sum_{i \in \mathcal{N}} x_i^{D*}$ to represent $\sum_{i \in \mathcal{N}}(p_i + D_i^{\text{Ad}})$ from (16). Primal feasibility is (3),(12),(14),(16),(17). Complementary slackness is constructed by the dual variables and the associated inequality constraints. We define the following dual variables for each constraint. μ is the dual variable of (16). $\bar{\kappa}$ and $\underline{\kappa}$ are the dual variables of the upper and lower bound constraints of (3). $\bar{\alpha}_l$ and $\underline{\alpha}_l$ are the dual variables for the branch flow limits in (17). $\bar{\lambda}_i$, $\underline{\lambda}_i$ and $\bar{\beta}_i$, $\underline{\beta}_i$ are the dual variables for (12) and (14). Stationarity condition is:

- w.r.t. $z^{\text{D-U}}$, p_i and D_i^{Ad},$i \in \mathcal{N}$:

$$\begin{cases} 0 \in \partial[P_{\text{buy}} \cdot \max(z^{\text{D-U}}, 0) + P_{\text{sell}} \cdot \min(z^{\text{D-U}}, 0)] \\ +\mu + \bar{\kappa} - \underline{\kappa} + \begin{cases} P_{\text{de}}, & \text{if } P_{\text{de}} \geq 0, P_{\text{in}} = 0; \\ -P_{\text{in}}, & \text{if } P_{\text{in}} \geq 0, P_{\text{de}} = 0, \end{cases} \\ \partial\tilde{C}_{g,i}^D(p_i) + \mu + \sum_{l=1}^E(\bar{\alpha}_l a^{(l,i)} - \underline{\alpha}_l a^{(l,i)}) + \bar{\lambda}_i - \underline{\lambda}_i \ni 0, \\ -\partial\tilde{B}_i(D_i^{\text{Ad}}) + \mu + \sum_{l=1}^E(\bar{\alpha}_l a^{(l,i)} - \underline{\alpha}_l a^{(l,i)}) + \bar{\beta}_i - \underline{\beta}_i \ni 0. \end{cases} \tag{49}$$

P3-D-R's objective is not continuous differentiable in $z^{\text{D-U}}$. The above (46) and (49) are KKT conditions in (sub)gradients [33], which are sufficient and necessary for convex optimizations. Note that the first term in (49) has $P_{\text{de}}/P_{\text{in}}$ as we use $\sum_{i \in \mathcal{N}}(D_i^* - x_i^{D*}) - z^{\text{D-U}}$ to represent $\sum_{i \in \mathcal{N}}(p_i + D_i^{\text{Ad}})$ in (31).

C.3 KKT conditions for DR-sub

Primal feasibility is (12), (14). Complementary slackness is constructed by the dual variables and the associated inequality constraints. Let $\bar{\varsigma}_i$, $\underline{\varsigma}_i$ and $\bar{\varkappa}_i$, $\underline{\varkappa}_i$ be the dual variables of (12) and (14). The stationarity condition is:

- w.r.t. p_i and D_i^{Ad}, $i \in \mathcal{N}$:

$$\begin{cases} \partial\tilde{C}_{g,i}^D(p_i) - P_i^{dp} + \bar{\varsigma}_i - \underline{\varsigma}_i \ni 0, \\ -\partial\tilde{B}_i(D_i^{\text{Ad}}) - P_i^{dp} + \bar{\varkappa}_i - \underline{\varkappa}_i \ni 0. \end{cases} \tag{50}$$

Observations:. Let $(z^{\text{U-G}*}, \mathbf{x}^{U*}, \mathbf{p}^*, \mathbf{D}^{\text{Ad}^*})$ be the unique optimum of **P3-U** and $(\eta^*, \bar{\zeta}^*, \underline{\zeta}^*, \bar{\theta}_j^*, \underline{\theta}_j^*, \bar{\epsilon}_l^*, \underline{\epsilon}_l^*, \bar{\gamma}_i^*, \underline{\gamma}_i^*, \bar{\tau}_i^*, \underline{\tau}_i^*)$ be its dual variables. Let $(\mu^*, \bar{\kappa}^*, \underline{\kappa}^*, \bar{\alpha}_l^*, \underline{\alpha}_l^*, \bar{\lambda}_i^*, \underline{\lambda}_i^*, \bar{\beta}_i^*, \underline{\beta}_i^*)$ be **P3-D-R**'s optimal dual variables. Let $(\bar{\varsigma}_i^*, \underline{\varsigma}_i^*, \bar{\varkappa}_i^*, \underline{\varkappa}_i^*)$ be **DR-sub**'s optimal dual variables. Note that the optimal dual variables may not be unique.

Consider a specific case that $z^{\text{D-U}*} > 0$ for better understanding, i.e., DSO imports energy from utility at **P3-U**'s optimum. Therefore, if $f'(z^{\text{U-G}*}) > P_{\text{buy}}$, by setting $P_{\text{de}} = f'(z^{\text{U-G}*}) - P_{\text{buy}}$, $P_{\text{in}} = 0$, the part primal optimum of **P3-U** $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}^*})$ satisfy the KKT conditions for **P3-D-R** by setting $(\mu^*, \bar{\kappa}^*, \underline{\kappa}^*, \bar{\alpha}_l^*, \underline{\alpha}_l^*, \bar{\lambda}_i^*, \underline{\lambda}_i^*, \bar{\beta}_i^*, \underline{\beta}_i^*) = (\eta^*, \bar{\zeta}^*, \underline{\zeta}^*, \bar{\epsilon}_l^*, \underline{\epsilon}_l^*, \bar{\gamma}_i^*, \underline{\gamma}_i^*, \bar{\tau}_i^*, \underline{\tau}_i^*)$. From (16), the optimal $z^{\text{D-U}*}$ can be uniquely recovered. Since conditions (49) are sufficient and necessary for **P3-D-R**, the optimal $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}^*})$ of **P3-U** is also optimal and unique to **P3-D-R**. Consider the KKT conditions of **DR-sub** then, if P_i^{dp} is set as Theorem 3, one can easily see that the part primal optimum of **P3-U** $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}^*})$ satisfy the KKT conditions for **DR-sub**, implying the equilibrium efficiency. The second and third claims can also be proved by the similar argument procedures in Lemma 2. In summary, for different scenarios in Theorem 2, the DR game has an efficient equilibrium that maximizes social welfare with the designed incentive price-setting scheme between utility and DSO, and the dynamic price-setting scheme between DSO and prosumers. In addition, we remark that the result hold for general convex subdifferentiable $C_{g,j}^U$, which still guarantees the unique $(\mathbf{p}^*, \mathbf{D}^{\text{Ad}^*})$ and hence maintain the efficient equilibrium. This completes the proof of Theorem 1, Theorem 2, and Theorem 3.

D PROOF OF THEOREM 5 AND THEOREM 6

D.1 Proof of Theorem 5

We sketch proof here. The unique optimal $(\mathbf{x}^{D*} + \mathbf{p}^*, \mathbf{D}^* - \mathbf{D}^{\text{Ad}*}, z^{\text{U-G}*}, \mathbf{x}^{U*})$ and the optimal incentive price are discussed in Sec. 5.3. For DSO's optimal strategy with false report, under (31), its KKT conditions are identical to **P3-D-R**'s. Hence, **P3-U**'s optimum is also optimal to DSO, implying the equilibrium efficiency and robustness under such a fault-ridden setting. In addition, suppose DSO reports a false $(\hat{\mathbf{x}}^D, \hat{\mathbf{D}}) \neq (\mathbf{x}^{D*}, \mathbf{D}^*)$ and $\sum_i (\hat{D}_i - \hat{x}_i^D) > \sum_i (D_i^* - x_i^{D*})$, i.e., more reported net load, though DSO may benefit with $P_{\text{de}} > 0$, it will suffer in the worst-case if a $P_{\text{in}} > 0$ is given. Similar result holds if less load is reported while P_{de} is provided. Hence, reporting truthfully maximizes DSO/prosumers' worst-case benefit.

D.2 Proof of Theorem 6

We first prove that the optimal objectives $(B_{\text{social}}^*, \tilde{B}_{\text{total}}^{D*}, B_{\text{total}}^{D*}, B_{\text{pro},i}^*, B_{\text{pro},i}^{DR*})$ of (**P2**,**P1**,**P0**,**P0-sub**,**DR-sub**) are concave functions of each $e_i, i \in \mathcal{N}$. Let us consider the optimal B_{social}^* for example. From Theorem 1, we know that the optimal objective of **P2** is the same as the convex optimization **P3-U**. Therefore, it is sufficient to consider **P3-U** for analysis. Let $B_{\text{social}}^*(\mathbf{e}_1)$ and $B_{\text{social}}^*(\mathbf{e}_2)$ are the optimal objectives of **P3-U** under two different $\mathbf{e}_1$ and $\mathbf{e}_2$, where $\mathbf{e}$ is a vector that has elements of $e_i, i \in \mathcal{N}$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the corresponding optimal decision variables. For any value $\lambda \in (0, 1)$ and $\hat{\mathbf{e}} = \lambda \cdot \mathbf{e}_1 + (1-\lambda) \cdot \mathbf{e}_2$, we have

$$B_{\text{social}}^*(\hat{\mathbf{e}}) \geq B_{\text{social}}(\hat{\mathbf{e}}, \lambda \cdot \mathbf{x}_1 + (1-\lambda) \cdot \mathbf{x}_2) \tag{51}$$

$$\geq \lambda \cdot B_{\text{social}}(\mathbf{e}_1, \mathbf{x}_1) + (1-\lambda) \cdot B_{\text{social}}(\mathbf{e}_2, \mathbf{x}_2), \tag{52}$$

where the first inequality comes from that $\lambda \cdot \mathbf{x}_1 + (1-\lambda) \cdot \mathbf{x}_2$ is feasible to optimization **P3-U** and therefore, the objective at $\lambda \cdot \mathbf{x}_1 + (1-\lambda) \cdot \mathbf{x}_2$ provides the lower bound to **P3-U**. The second inequality is from the concavity of the function B_{social}. Therefore, we conclude that the optimal B_{social}^* is concave w.r.t. $\mathbf{e}$. Similar results hold for $(B_{\text{pro},i}^*, B_{\text{pro},i}^{DR*}, B_{\text{total}}^{D*}, \tilde{B}_{\text{total}}^{D*})$. For the expectation of concave function to mean-preserving spread variables $\mathbf{e}^a$ and $\mathbf{e}^b$ with variance $\sigma_a^2 \geq \sigma_b^2$, it is direct from [38, 47] that

$$\mathbb{E}[B_{\text{social}}(\mathbf{e}^a)] \leq \mathbb{E}[B_{\text{social}}(\mathbf{e}^b)]. \tag{53}$$

The concavity of the other optimal objective functions $(\tilde{B}_{\text{total}}^{D*}, B_{\text{total}}^{D*}, B_{\text{pro},i}^*, B_{\text{pro},i}^{DR*})$ can be proved by the similar argument, e.g., $\tilde{B}_{\text{total}}^{D*}$ is concave as the equivalent optimal objective of program **P3-D** is concave. This concludes the proof of Theorem 6.